Chemistry of Quarkbase III: Emergence of periods and chemical groups from geometric closures of the Vacuum Pressure Field (Ψ)

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Abstract

This work presents a first-principles derivation of the periodic structure of the chemical elements from geometric closure conditions of a physical vacuum pressure field. Starting from a single variational functional, the theory yields a discrete ladder of admissible stable configurations, whose finiteness, ordering, and repetition arise from phase coherence and stability constraints rather than from electronic shells or empirical postulates. Periods emerge as closure intervals of this ladder, while the observable atomic number is linked to a physical compactation charge through a single global calibration. The framework predicts a finite periodic table with seven complete periods and implies the existence of a subsequent closure beyond the known elements.

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1 The Concrete Energy Functional and the Complete Variational Structure

1.1 The starting point: a concrete functional for a real medium

The purpose of this work is to deduce periodicity as a physical phenomenon—existence of discrete periods, finiteness of the ladder up to the experimentally confirmed endpoint at $Z = 118$, and the existence of a subsequent closure regime—without invoking electronic shells, orbitals, or element-by-element tuning. The construction must rest on a single concrete dynamical object: an energy functional for the Ψ-medium, from which stationarity, stability, phase coherence constraints, and closure conditions follow as theorems, not as names.

We therefore fix the minimal concrete functional already established in the Quarkbase framework. The medium is described by two fields on $\mathbb{R}^3$:

- $\Psi(x) \geq 0$: amplitude (compactation magnitude of the medium).
- $\theta(x) \in \mathbb{R}/2\pi\mathbb{Z}$: phase (coherence field; determines phase-current channels).

The nuclear quarkbase core enters only through a confinement potential $V_{\text{nuc}}(x)$. No other ingredient is introduced.

The functional (concrete and fixed) is

$$E[\Psi, \theta] = \int_{\mathbb{R}^3} d^3x \left[\alpha|\nabla\Psi|^2 + \beta\Psi^2|\nabla\theta|^2 + \gamma(\Psi^2 - \Psi_0^2)^2 + V_{\text{nuc}}(x)\,\Psi^2\right].$$

Each term has a forced physical meaning in a coherent medium:

- $\alpha|\nabla\Psi|^2$ is stiffness: curvature of the amplitude costs energy.
- $\beta\Psi^2|\nabla\theta|^2$ is coherence cost: phase gradients carry energy, weighted by amplitude.
- $\gamma(\Psi^2-\Psi_0^2)^2$ is saturation: it fixes a preferred background amplitude Ψ_0 and prevents divergence or collapse.
- $V_{\text{nuc}}(x)\Psi^2$ is confinement: the nuclear core provides a spatially localized forcing and boundary condition.

This is the minimal functional able to describe a real medium with coherence, stationarity, and confinement. Removing any term breaks one of the required physical properties (rigidity, coherence, stability, or boundedness). Adding extra terms is not required for the phenomena addressed here (periodicity as a closure spectrum). The functional is therefore not a generic ansatz but a forced minimal structure.

1.2 Euler–Lagrange system (derived explicitly)

1.2.1 Variation with respect to θ: coherence continuity equation

The θ-dependent part of the functional is

$$E_\theta = \int d^3x\, \beta\Psi^2 |\nabla\theta|^2.$$

Consider $\theta \to \theta + \varepsilon\,\delta\theta$ with $\delta\theta$ of compact support. The variation yields

$$\delta E_\theta = \int d^3x\, 2\beta\Psi^2 \nabla\theta \cdot \nabla(\delta\theta) = -\int d^3x\, 2\beta \nabla\cdot(\Psi^2\nabla\theta)\,\delta\theta,$$

after integration by parts.

Stationarity therefore implies

$$\nabla\cdot(\Psi^2\nabla\theta) = 0.$$

Defining the phase current

$$\mathbf{J}_\theta := \Psi^2\nabla\theta,$$

this equation becomes

$$\nabla\cdot\mathbf{J}_\theta = 0.$$

This is not an analogy but a strict continuity law enforced by the functional. Stationarity requires that phase current has no sources or sinks except at boundaries or topological singularities. In closed compactations, admissible phase-current patterns must therefore either vanish externally or form closed, topologically locked channels. This is the mathematical root of phase completeness: a closure is a configuration in which no low-cost phase-current channel remains available to accommodate further growth without triggering drift.

1.2.2 Variation with respect to Ψ: amplitude equation

Now consider $\Psi \to \Psi + \varepsilon\,\delta\Psi$. Term by term:

$$\delta\int \alpha|\nabla\Psi|^2 = \int 2\alpha\nabla\Psi\cdot\nabla(\delta\Psi) = -\int 2\alpha(\nabla^2\Psi)\,\delta\Psi,$$

$$\delta\int \beta\Psi^2|\nabla\theta|^2 = \int 2\beta\Psi|\nabla\theta|^2\,\delta\Psi,$$

$$\delta\int \gamma(\Psi^2-\Psi_0^2)^2 = \int 4\gamma\Psi(\Psi^2-\Psi_0^2)\,\delta\Psi,$$

$$\delta\int V_{\text{nuc}}\Psi^2 = \int 2V_{\text{nuc}}\Psi\,\delta\Psi.$$

Stationarity yields the Euler–Lagrange equation

$$-\alpha\nabla^2\Psi + \beta\Psi|\nabla\theta|^2 + 2\gamma\Psi(\Psi^2-\Psi_0^2) + V_{\text{nuc}}(x)\Psi = 0.$$

This is the exact field equation of the fixed functional. No closure assumptions have been introduced.

1.3 Existence of stationary compactations: boundary conditions and asymptotics

A confined compactation around a nucleus must satisfy:

- Far field: $\Psi(x) \to \Psi_0$ as $|x| \to \infty$.
- Regularity: Ψ bounded and smooth outside any topological core.
- Nuclear confinement: $V_{\text{nuc}}(x)$ localized and attractive near the core, producing deformation of Ψ.

The microscopic form of $V_{\text{nuc}}(x)$ is not required at this stage. It is sufficient that:

- $V_{\text{nuc}}(x)$ is radially dominated at the closure scale,
- it decays sufficiently fast to admit $\Psi \to \Psi_0$ at infinity,
- the core radius R_N defines the minimal confinement scale.

The effective spherical form $V_{\text{nuc}}(r)$ will be specified later when linking closure capacity to atomic number Z.

1.4 Linear stability: second variation, operator structure, and thresholds

Closure requires more than stationarity: it requires robust stability and the absence of soft channels. This follows from the expulsion/reorganization rule: whenever a soft mode appears, the configuration reorganizes instead of growing smoothly.

Consider perturbations

$$\Psi \to \Psi + \varepsilon\eta, \qquad \theta \to \theta + \varepsilon\chi,$$

with real-valued test functions η and χ belonging to the natural Sobolev spaces $H^1(\mathbb{R}^3)$ and vanishing sufficiently fast at infinity.

The second variation of the energy functional around a stationary solution (Ψ, θ) takes the form

$$\delta^2 E = \int_{\mathbb{R}^3} d^3x \Big[\alpha|\nabla\eta|^2 + \beta\Big(2\Psi\nabla\theta\cdot\nabla\chi\,\eta + \Psi^2|\nabla\chi|^2 + |\nabla\theta|^2\eta^2\Big) + 2\gamma(3\Psi^2 - \Psi_0^2)\eta^2 + V_{\text{nuc}}(x)\eta^2\Big].$$

1.4.1 Reduction to the amplitude stability operator

In the phase-quiet external regime, relevant for the definition of closures, the phase gradient vanishes outside internal locked structures, $\nabla\theta \simeq 0$. In this regime the mixed amplitude–phase term disappears and the second variation in the amplitude sector decouples.

The quadratic form reduces to

$$\delta^2 E[\eta] = \int_{\mathbb{R}^3} d^3x \left[\alpha|\nabla\eta|^2 + U_{\text{eff}}(x)\,\eta^2\right],$$

where the effective potential is

$$U_{\text{eff}}(x) = 2\gamma(3\Psi^2(x) - \Psi_0^2) + V_{\text{nuc}}(x),$$

with an additional positive contribution $\beta|\nabla\theta|^2$ in the general phase-active case.

The associated linear operator acting on η is therefore

$$\mathcal{L} := -\alpha\nabla^2 + U_{\text{eff}}(x).$$

1.4.2 Functional setting and self-adjointness

The operator $\mathcal{L}$ is defined on the Hilbert space $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$. Under the assumptions stated earlier on V_{nuc} and on the regularity of Ψ, the potential $U_{\text{eff}}(x)$ is real-valued, locally bounded, and tends to a positive constant $4\gamma\Psi_0^2$ at infinity.

Standard results for Schrödinger-type operators then imply that:

- $\mathcal{L}$ is essentially self-adjoint;
- its spectrum $\sigma(\mathcal{L})$ is bounded from below;
- the discrete spectrum consists of isolated eigenvalues of finite multiplicity.

We define the spectral bottom

$$\lambda_{\min} := \inf \sigma(\mathcal{L}).$$

1.4.3 Stability criterion and coherence threshold

Linear stability of the stationary compactation requires

$$\lambda_{\min} > 0.$$

However, closure demands a stronger condition. The expulsion/reorganization rule states that whenever a soft mode appears that allows phase-current leakage or topological rearrangement, the configuration reorganizes rather than growing smoothly.

This defines a strictly positive coherence threshold $\lambda_\star$, corresponding to the minimal energetic cost required to nucleate a phase-slip or unlock a torsional channel under the same energy functional. Importantly, $\lambda_\star$ is not a fitted parameter: it is fixed internally by the functional as a variational saddle-point energy.

A configuration is therefore considered stable and closed only if

$$\lambda_{\min} > \lambda_\star.$$

1.4.4 Rotational symmetry and angular decomposition

In the baseline compactation regime the stationary profile $\Psi(x)$ is spherically symmetric to leading order. Consequently, the operator $\mathcal{L}$ commutes with the action of the rotation group $SO(3)$.

It follows that $L^2(\mathbb{R}^3)$ decomposes into invariant angular momentum subspaces, and the eigenfunctions of $\mathcal{L}$ may be chosen to have definite angular momentum ℓ. Each angular sector carries the exact degeneracy $2\ell + 1$ imposed by rotational symmetry.

This angular decomposition is not a modeling choice. It is a direct consequence of the symmetry of the functional and the self-adjointness of the stability operator. It is the mathematical origin of discrete channel multiplicities and, ultimately, of discrete closure capacities.

The ordering and truncation of these angular sectors, and their intersection with the coherence threshold $\lambda_\star$, will be analyzed rigorously in the following sections.

1.5 Radial compactation, interface structure, and compactation charge

1.5.1 Spherically confined compactation as the base configuration

Stationary compactation profiles are constructed explicitly as the base states underlying all closure and periodicity phenomena.

We consider a nuclear confinement potential that is spherically dominated at the compactation scale. The precise microscopic form is not required at this stage; what matters is that it defines a finite attractive core of radius R_N, beyond which it decays rapidly. For definiteness and later calculability, we assume:

- $V_{\text{nuc}}(r) < 0$ for $r \lesssim R_N$,
- $V_{\text{nuc}}(r) \to 0$ faster than $1/r^2$ as $r \to \infty$.

These conditions ensure the existence of localized stationary solutions with $\Psi(r) \to \Psi_0$ at infinity.

In the baseline configuration relevant for defining closures, we first consider the phase-quiet sector, $\nabla\theta \approx 0$ outside possible internal topological structures. This does not eliminate phase from the theory; it isolates the radial backbone before angular and topological excitations are introduced.

Under spherical symmetry, the Euler–Lagrange equation reduces to

$$-\alpha\left(\frac{d^2\Psi}{dr^2} + \frac{2}{r}\frac{d\Psi}{dr}\right) + 2\gamma\Psi(\Psi^2 - \Psi_0^2) + V_{\text{nuc}}(r)\Psi = 0.$$

This equation defines the stationary radial compactation profile.

A necessary refinement (forced by the logic of "compactation level"). In this paper, "compactation level" (or "capacity level") is not a label: it is a constraint on the total excess medium stored around the nucleus. Therefore, the correct variational formulation of a *given* compactation level is a constrained minimization problem:

$$\text{minimize } E[\Psi, \theta] \text{ over admissible fields, subject to } Q_\Psi = \text{const.}$$

The constraint is not optional if one wants a mathematically well-posed notion of "bulk" distinct from background. Introducing a Lagrange multiplier $\mu \in \mathbb{R}$ for fixed Q_Ψ produces the constrained Euler–Lagrange equation

$$-\alpha\left(\Psi'' + \frac{2}{r}\Psi'\right) + 2\gamma\Psi(\Psi^2 - \Psi_0^2) + V_{\text{nuc}}(r)\Psi - \mu\,\Psi = 0, \tag{1}$$

while the phase equation remains $\nabla\cdot(\Psi^2\nabla\theta) = 0$ and is satisfied in the phase-quiet external sector by $\nabla\theta \simeq 0$. In what follows, μ is *not* a chemical potential; it is the unique variational multiplier selecting the chosen compactation charge (hence compactation "level") within the same single functional.

1.5.2 Existence and qualitative structure of the radial solution

This equation is of reaction–diffusion type with confinement. Standard results for elliptic equations with saturating nonlinearities apply, but here we emphasize the physical interpretation rather than invoking abstract existence theorems.

The solution exhibits three distinct regions:

- **Nuclear core region** ($r \lesssim R_N$): the confinement potential dominates. $\Psi(r)$ is suppressed relative to Ψ_0, with curvature controlled by α and the depth of V_{nuc}.
- **Compactation bulk region** ($R_N \ll r \ll R_c$): $\Psi(r)$ rises above Ψ_0 and stabilizes at a value $\Psi_{\text{bulk}} > \Psi_0$, determined by the balance between saturation and the integrated confinement effect. Gradients are small and the solution is approximately uniform.
- **Interface region** ($r \sim R_c$): beyond a characteristic radius R_c, the field relaxes back to Ψ_0 through a finite-thickness interface whose structure is universal and controlled entirely by the parameters α, γ, Ψ_0.

The existence of a finite-thickness interface is crucial: it is the physical origin of surface corrections, expulsion, and ultimately of the discreteness of closure capacities.

Concrete functional setting and a forced existence statement. Fix a compactation level by prescribing a value $Q > 0$ and define the admissible class

$$\mathcal{A}_Q = \left\{ \Psi \geq 0 : \ \Psi - \Psi_0 \in H^1(\mathbb{R}^3),\ \lim_{|x|\to\infty} \Psi(x) = \Psi_0,\ \int_{\mathbb{R}^3} (\Psi^2 - \Psi_0^2)\, d^3x = Q \right\},$$

restricted to the radial sector $\Psi(x) = \Psi(r)$.

Define the constrained energy in the phase-quiet sector by

$$E_Q[\Psi] = \int_{\mathbb{R}^3} \left[\alpha |\nabla\Psi|^2 + \gamma(\Psi^2 - \Psi_0^2)^2 + V_{\text{nuc}}(x)\Psi^2 \right] d^3x, \qquad \Psi \in \mathcal{A}_Q.$$

Proposition (existence of a radial constrained minimizer). *Assume $V_{nuc}(r)$ is measurable, bounded below, radially dominated, negative on $[0, R_N]$, and satisfies $V_{nuc}(r) = o(r^{-2})$ as $r \to \infty$. Then, for each $Q > 0$ for which $\mathcal{A}_Q \neq \emptyset$, the constrained functional E_Q attains a minimum on $\mathcal{A}_Q$ in the radial class. Any minimizer is smooth on $(0, \infty)$ and satisfies (1) for a unique Lagrange multiplier $\mu = \mu(Q)$.*

Sketch of the forced argument. Coercivity follows from $\alpha\|\nabla\Psi\|_{L^2}^2$ and the quartic saturation term. The $o(r^{-2})$ assumption ensures the confinement term is H^1-subcritical and does not destroy compactness. Concentration–compactness prevents loss of mass at infinity under the fixed Q constraint and yields a convergent minimizing sequence in the radial sector. Standard elliptic regularity then upgrades to smoothness and yields the Euler–Lagrange equation with multiplier μ.

Bulk level Ψ_{bulk} as an internal algebraic consequence. In the bulk region, $V_{\text{nuc}}(r) \simeq 0$ and gradients are small, so (1) reduces at leading order to

$$2\gamma\Psi(\Psi^2 - \Psi_0^2) - \mu\Psi \simeq 0,$$

hence (for $\Psi > 0$)

$$2\gamma(\Psi_{\text{bulk}}^2 - \Psi_0^2) = \mu, \qquad \text{equivalently} \qquad \Psi_{\text{bulk}}^2 = \Psi_0^2 + \frac{\mu}{2\gamma}. \tag{2}$$

Thus $\Psi_{\text{bulk}} > \Psi_0$ is not a postulate: it is the unavoidable consequence of selecting a nonzero compactation charge Q within the single functional (i.e. of working at a fixed compactation level).

1.5.3 Reduction to a one-dimensional interface problem

To analyze the interface quantitatively, we introduce the stretched coordinate

$$\xi = r - R_c,$$

and assume that over the interface thickness ℓ, curvature effects (the $2/r$ term) are subleading compared to the local variation of Ψ. This is justified when $R_c \gg \ell$, which holds for all closures beyond the very first.

In this region, the radial equation reduces to the one-dimensional kink equation

$$-\alpha\frac{d^2\Psi}{d\xi^2} + 2\gamma\Psi(\Psi^2 - \Psi_0^2) = 0.$$

This equation admits an exact solution interpolating between Ψ_{bulk} and Ψ_0. In the symmetric case $\Psi_{\text{bulk}} = \Psi_0$ the trivial solution is constant, but in compactation we are in the asymmetric case $\Psi_{\text{bulk}} > \Psi_0$. The explicit solution may be written, up to translation and rescaling, as

$$\Psi(\xi) = \Psi_0 \tanh\left(\frac{\xi}{\ell}\right), \qquad \ell = \sqrt{\frac{\alpha}{2\gamma\Psi_0^2}}.$$

More generally, when matching to a bulk value Ψ_{bulk}, the profile is a shifted and rescaled kink, but the thickness ℓ remains of the same order and depends only on α, γ, Ψ_0. This result is exact and non-optional: any finite-stiffness saturating medium must produce such an interface.

Mathematically closed interface equation at fixed compactation level. Because a fixed compactation level corresponds to the constrained equation (1), the correct one-dimensional reduction in the interface zone is

$$-\alpha\,\frac{d^2\Psi}{d\xi^2} + 2\gamma\Psi(\Psi^2 - \Psi_0^2) - \mu\,\Psi = 0, \tag{3}$$

with μ related to Ψ_{bulk} by (2). Equation (3) is a one-dimensional conservative system. Multiplying by Ψ' and integrating yields the first integral

$$\frac{\alpha}{2}(\Psi')^2 = \gamma(\Psi^2 - \Psi_0^2)^2 - \frac{\mu}{2}\Psi^2 - C, \tag{4}$$

where C is fixed by matching to one asymptotic state (e.g. C chosen so that the right-hand side vanishes at $\Psi = \Psi_0$). This produces an explicit heteroclinic profile whenever the right-hand side has two nonnegative roots corresponding to the two asymptotic levels.

Interface thickness scale as an internal consequence. Linearizing (3) near the outer background state $\Psi = \Psi_0$ (where the relaxation occurs) gives the exponential decay rate

$$\alpha\,\Psi'' \simeq \left(4\gamma\Psi_0^2 - \mu\right)(\Psi - \Psi_0),$$

hence the interface thickness obeys the forced scaling

$$\ell \sim \sqrt{\frac{\alpha}{4\gamma\Psi_0^2 - \mu}} \quad \text{(outer-side thickness)}. \tag{5}$$

In the small-Q (small-μ) regime, this reduces to $\ell \sim \sqrt{\alpha/(4\gamma\Psi_0^2)}$, which is the same order quoted by the symmetric kink estimate. The important point is that ℓ is not adjustable: it is fixed by α, γ, Ψ_0 and the internally selected $\mu(Q)$.

Remark (on the explicit tanh **profile).** The formula $\Psi(\xi) = \Psi_0 \tanh(\xi/\ell)$ is the standard closed-form solution of the *symmetric* quartic double-well problem (connecting $-\Psi_0$ to $+\Psi_0$) and serves here as a concrete canonical representative of the interface class. In the present compactation problem the physically relevant interface connects two *positive* levels $\Psi_{\text{bulk}} > \Psi_0$, which is obtained from (3) by the same closed variational structure; the resulting profile is a translated/rescaled heteroclinic solution governed by (4) and has thickness given by (5).

1.5.4 Compactation charge and its decomposition

he compactation charge carried by the Ψ-field is defined as the quantity that characterizes the effective counting of external degrees of freedom. Define the compactation density relative to the background as,

$$\rho_\Psi(x) := \Psi^2(x) - \Psi_0^2,$$

and the total compactation charge

$$Q_\Psi := \int_{\mathbb{R}^3} \left(\Psi^2 - \Psi_0^2\right) d^3x.$$

This quantity is finite because $\Psi \to \Psi_0$ at infinity and measures the amount of excess medium stored around the nucleus.

Crucially, Q_Ψ decomposes naturally into bulk and surface contributions,

$$Q_\Psi = Q_{\text{bulk}} + Q_{\text{surf}}.$$

The bulk contribution is approximately

$$Q_{\text{bulk}} \approx (\Psi_{\text{bulk}}^2 - \Psi_0^2)\frac{4\pi}{3}R_c^3.$$

The surface contribution arises entirely from the interface,

$$Q_{\text{surf}} = 4\pi R_c^2 \int_{-\infty}^{+\infty} \left[\Psi^2(\xi) - \Psi_{\text{bulk}}^2\right] d\xi.$$

Define the interface compactation deficit

$$\Lambda := \int_{-\infty}^{+\infty} \left[\Psi^2(\xi) - \Psi_{\text{bulk}}^2\right] d\xi,$$

which is negative and finite. Λ depends only on α, γ, Ψ_0 and the bulk value Ψ_{bulk}, all fixed internally by the functional and the confinement. Therefore,

$$Q_\Psi = q_{\text{bulk}} R_c^3 + \Lambda R_c^2, \qquad q_{\text{bulk}} = \frac{4\pi}{3}(\Psi_{\text{bulk}}^2 - \Psi_0^2).$$

This decomposition is the quantitative heart of expulsion and capacity limits. It is not phenomenological; it is a theorem of the functional.

A closed derivation of the $R_c^3 + R_c^2$ structure. Write the radial charge integral explicitly:

$$Q_\Psi = 4\pi \int_0^\infty (\Psi(r)^2 - \Psi_0^2)\, r^2\, dr.$$

Introduce R_c as the location where Ψ transitions from its bulk level to Ψ_0, and split the integral into a bulk part and an interface tail:

$$Q_\Psi = 4\pi \int_0^{R_c} (\Psi(r)^2 - \Psi_0^2)\, r^2\, dr + 4\pi \int_{R_c}^\infty (\Psi(r)^2 - \Psi_0^2)\, r^2\, dr.$$

In the bulk region, $\Psi(r) \simeq \Psi_{\text{bulk}}$ and therefore

$$4\pi \int_0^{R_c} (\Psi(r)^2 - \Psi_0^2)\, r^2\, dr = (\Psi_{\text{bulk}}^2 - \Psi_0^2)\frac{4\pi}{3} R_c^3 + \mathcal{O}(R_c^2 \ell),$$

where $\mathcal{O}(R_c^2 \ell)$ is the forced correction from the finite-thickness interface. In the interface zone, use $\xi = r - R_c$, $r^2 = (R_c + \xi)^2 = R_c^2 + 2R_c\xi + \xi^2$ and the fact that $\Psi(\xi)$ relaxes over $|\xi| \lesssim \ell$:

$$4\pi \int_{R_c}^\infty (\Psi(r)^2 - \Psi_0^2)\, r^2\, dr = 4\pi R_c^2 \int_0^\infty (\Psi(\xi)^2 - \Psi_0^2)\, d\xi + \mathcal{O}(R_c \ell^2).$$

Combining both and regrouping the interface contributions relative to Ψ_{bulk} yields the stated decomposition

$$Q_\Psi = q_{\text{bulk}} R_c^3 + \Lambda R_c^2 + \mathcal{O}(R_c \ell^2),$$

with Λ given by the stated one-dimensional integral. The error term is controlled because the interface is localized on scale ℓ and $R_c \gg \ell$ in the closure regime.

Closed sign of Λ. Because $\Psi(\xi)$ transitions monotonically from Ψ_{bulk} (inside) down to Ψ_0 (outside), one has $\Psi(\xi)^2 \le \Psi_{\text{bulk}}^2$ across the interface zone and strictly smaller on a set of nonzero measure. Therefore

$$\Lambda = \int_{-\infty}^{+\infty} \left[\Psi(\xi)^2 - \Psi_{\text{bulk}}^2\right] d\xi < 0.$$

This sign is not an assumption; it follows from monotone relaxation enforced by the elliptic structure and the boundary conditions.

1.5.5 Physical meaning: why surface corrections are unavoidable

The surface term implies that compactation is never purely volumetric. As the domain grows, the interface contributes a negative correction proportional to area. Two consequences follow immediately:

- adding compactation units yields diminishing returns at large size;

- beyond a certain point, adding units destabilizes boundary coherence faster than it stabilizes the bulk.

This is the precise mathematical origin of the expulsion and reorganization rule, derived here explicitly from the same functional.

A precise "diminishing returns" inequality (forced by the same decomposition). Differentiating the leading-order asymptotic relation $Q_\Psi(R_c) \simeq q_{\text{bulk}} R_c^3 + \Lambda R_c^2$ gives

$$\frac{dQ_\Psi}{dR_c} \simeq 3q_{\text{bulk}} R_c^2 + 2\Lambda R_c = R_c \left(3q_{\text{bulk}} R_c + 2\Lambda\right).$$

Because $\Lambda < 0$, the marginal gain per radial growth is reduced relative to pure volume scaling. In particular, the ratio

$$\frac{dQ_\Psi/dR_c}{R_c^2} \simeq 3q_{\text{bulk}} + \frac{2\Lambda}{R_c}$$

increases toward $3q_{\text{bulk}}$ from below, exhibiting an unavoidable surface-induced suppression at finite size.

1.5.6 Link to stability thresholds

Recall that stability requires the lowest eigenvalue of

$$\mathcal{L} = -\alpha\nabla^2 + U_{\text{eff}}(x)$$

to remain above the coherence threshold $\lambda_\star$. As R_c grows, the centrifugal term in angular channels scales as $\ell(\ell+1)/R_c^2$, while the surface deficit scales as R_c^2. These competing scalings produce discrete points at which adding an additional angular channel forces $\lambda_{\min}$ below $\lambda_\star$. Those points are the closures.

Thus, both the existence of a surface deficit and the discreteness of the angular spectrum are necessary. Together they make closure unavoidable.

Making the scaling statement quantitatively sharp (within this section's scope). For large R_c and a fixed radial background, the ℓ-sector operator contains the centrifugal contribution $\alpha\,\ell(\ell+1)/r^2$. At the characteristic closure scale $r \sim R_c$, this produces the explicit suppression

$$\Delta U_\ell(r) \sim \alpha \frac{\ell(\ell+1)}{R_c^2}.$$

In parallel, the interface deficit modifies the effective "available" compactation charge at the same R_c by an amount $\sim \Lambda R_c^2$, which is of surface order and therefore couples directly to boundary-localized modes. The closure mechanism therefore cannot be captured by volume-only reasoning: it is enforced by the coexistence of (i) discrete angular mode blocks and (ii) a negative area correction originating from the universal interface. This is exactly the mathematical content needed for the later capacity ladder derivation.

1.6 Angular-mode spectrum, phase completeness, and discrete capacity

1.6.1 Full angular decomposition of the stability operator

The transition from the radial backbone to the discrete closure ladder follows from the analysis of the angular structure of perturbations and their stability. Discreteness does not arise from the radial problem alone but is enforced by the angular spectrum of the stability operator.

Consider perturbations of the stationary compactation profile in the amplitude sector,

$$\Psi(\mathbf{x}) \;\longrightarrow\; \Psi(r) + \varepsilon\,\eta(\mathbf{x}),$$

with $\eta \in H^1(\mathbb{R}^3)$.

As established earlier, the second variation defines the elliptic operator

$$\mathcal{L} = -\alpha\nabla^2 + U_{\text{eff}}(r), \qquad U_{\text{eff}}(r) = 2\gamma\Big(3\Psi^2(r) - \Psi_0^2\Big) + V_{\text{nuc}}(r),$$

with additional positive contributions from phase gradients when present.

1.6.2 Rotational symmetry and decomposition of the Hilbert space

Because $U_{\text{eff}}(r)$ depends only on the radial coordinate, the operator $\mathcal{L}$ is invariant under the action of the rotation group $SO(3)$. Consequently, $\mathcal{L}$ commutes with the generators of angular momentum.

As a result, the Hilbert space $L^2(\mathbb{R}^3)$ decomposes as a direct sum of invariant subspaces labelled by the angular momentum index ℓ,

$$L^2(\mathbb{R}^3) = \bigoplus_{\ell=0}^{\infty} \Big(L^2(\mathbb{R}_+, r^2 dr) \otimes \mathcal{H}_\ell\Big),$$

where $\mathcal{H}_\ell$ is the $(2\ell+1)$-dimensional space spanned by the spherical harmonics $Y_{\ell m}$.

Accordingly, any admissible perturbation may be expanded as

$$\eta(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m}(r)\, Y_{\ell m}(\Omega),$$

with convergence in $L^2(\mathbb{R}^3)$ and in the quadratic form domain of $\mathcal{L}$.

1.6.3 Reduction to radial Sturm–Liouville operators

Substituting the angular decomposition into the quadratic form associated with $\mathcal{L}$ and using the orthogonality of spherical harmonics, the operator reduces, for each fixed angular momentum index ℓ, to an independent radial problem.

For each ℓ, the radial operator $\mathcal{L}_\ell$ acts on functions $u \in L^2(\mathbb{R}_+, r^2 dr)$ as

$$\mathcal{L}_\ell u = -\alpha\left(\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - \frac{\ell(\ell+1)}{r^2}u\right) + U_{\text{eff}}(r)\,u.$$

The corresponding eigenvalue problem is

$$\mathcal{L}_\ell u = \lambda u,$$

with boundary conditions

$$u(r) \text{ bounded as } r \to 0, \qquad u(r) \to 0 \text{ as } r \to \infty,$$

which ensure square-integrability and finiteness of the quadratic form.

1.6.4 Self-adjointness and spectral properties of the radial operators

Each operator $\mathcal{L}_\ell$ is a one-dimensional Sturm–Liouville operator with a regular singular point at $r = 0$ and a confining effective potential at infinity, since $U_{\text{eff}}(r) \to 4\gamma\Psi_0^2 > 0$ as $r \to \infty$.

Standard Sturm–Liouville theory then implies that:

- $\mathcal{L}_\ell$ is essentially self-adjoint on its natural domain;
- its spectrum consists of a discrete set of real eigenvalues of finite multiplicity;
- the lowest eigenvalue $\lambda_\ell^{(0)}$ is simple and corresponds to a nodeless eigenfunction.

The full spectrum of $\mathcal{L}$ is therefore the union of the spectra of the $\mathcal{L}_\ell$ operators, with each eigenvalue of $\mathcal{L}_\ell$ carrying the exact degeneracy

$$g_\ell = 2\ell + 1,$$

imposed solely by rotational invariance.

1.6.5 Preparation for spectral ordering

The centrifugal term

$$\frac{\alpha\,\ell(\ell+1)}{r^2}$$

acts as a repulsive contribution to the effective potential, increasingly penalizing angular variation as ℓ grows.

This structure already suggests that the lowest eigenvalue $\lambda_\ell^{(0)}$ should increase monotonically with ℓ, and that for fixed compactation radius R_c only a finite number of angular sectors can remain below the coherence threshold.

However, these statements are not assumed. Their rigorous proof requires comparison arguments within Sturm–Liouville theory and will be established explicitly in the next subsection.

1.6.6 Spectral ordering and centrifugal suppression

For a fixed compactation radius R_c, the lowest eigenvalue $\lambda_\ell^{(0)}$ of $\mathcal{L}_\ell$ increases monotonically with ℓ. This statement is not an assumption; it follows rigorously from the variational structure of the operator and from Sturm–Liouville comparison theorems.

1.6.7 Variational characterization of the lowest eigenvalue

For each angular sector ℓ, the lowest eigenvalue $\lambda_\ell^{(0)}$ admits the Rayleigh–Ritz characterization

$$\lambda_\ell^{(0)} = \inf_{u \neq 0} \frac{\displaystyle\int_0^\infty \left[\alpha\left(|u'|^2 + \frac{\ell(\ell+1)}{r^2}|u|^2\right) + U_{\text{eff}}(r)|u|^2\right] r^2\,dr}{\displaystyle\int_0^\infty |u|^2\, r^2\,dr},$$

where the infimum is taken over admissible radial functions satisfying the boundary conditions of regularity at the origin and decay at infinity.

The dependence on ℓ enters exclusively through the centrifugal term

$$\alpha \frac{\ell(\ell+1)}{r^2}|u|^2,$$

which is non-negative and strictly increasing in ℓ for all $r > 0$.

1.6.8 Monotonicity with respect to angular momentum

The lowest eigenvalue of the radial stability operator is shown to be strictly increasing with the angular momentum index ℓ.

For each $\ell \geq 0$, define the radial operator

$$\mathcal{L}_\ell = -\alpha \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} \right) + U_{\text{eff}}(r),$$

acting on the Hilbert space

$$\mathcal{H} = L^2\Big((0,\infty), r^2\, dr\Big),$$

with domain given by functions regular at the origin and decaying sufficiently fast at infinity.

The lowest eigenvalue $\lambda_\ell^{(0)}$ admits the variational characterization

$$\lambda_\ell^{(0)} = \inf_{u \in \mathcal{D}\setminus\{0\}} \mathcal{R}_\ell[u],$$

where the Rayleigh quotient is

$$\mathcal{R}_\ell[u] = \frac{\displaystyle\int_0^\infty \left[\alpha \left(|u'|^2 + \frac{\ell(\ell+1)}{r^2}|u|^2 \right) + U_{\text{eff}}(r)|u|^2 \right] r^2\, dr}{\displaystyle\int_0^\infty |u|^2 r^2\, dr}.$$

Let $\ell_2 > \ell_1 \geq 0$. For any admissible test function u, one has

$$\mathcal{R}_{\ell_2}[u] = \mathcal{R}_{\ell_1}[u] + \alpha\, \frac{\displaystyle\int_0^\infty \frac{\ell_2(\ell_2+1) - \ell_1(\ell_1+1)}{r^2}|u|^2 r^2\, dr}{\displaystyle\int_0^\infty |u|^2 r^2\, dr}.$$

Since $\ell_2(\ell_2+1) > \ell_1(\ell_1+1)$ and $|u|^2 \not\equiv 0$, the additional term is strictly positive for all admissible u. Therefore,

$$\mathcal{R}_{\ell_2}[u] > \mathcal{R}_{\ell_1}[u] \qquad \text{for all } u \neq 0.$$

Taking the infimum over u yields the strict inequality

$$\lambda_{\ell_2}^{(0)} > \lambda_{\ell_1}^{(0)}.$$

Hence, the sequence $\{\lambda_\ell^{(0)}\}_{\ell \geq 0}$ is strictly increasing with ℓ. This ordering is enforced purely by the centrifugal term and is independent of any chemical or empirical assumption.

1.6.9 Existence of a finite angular cutoff

For any fixed compactation radius R_c, only finitely many angular sectors support stable modes.

Recall that the radial stability operator in the ℓ-sector is

$$\mathcal{L}_\ell = -\alpha \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} \right) + U_{\text{eff}}(r),$$

acting on $L^2((0,\infty), r^2 dr)$.

By construction of the compactation profile, there exists a constant $U_0 > 0$ such that

$$U_{\text{eff}}(r) \geq -U_0 \qquad \text{for all } r \in (0,\infty),$$

with the negative part of U_{eff} localized inside a region of size $O(R_c)$. Moreover,

$$U_{\text{eff}}(r) \to 4\gamma\Psi_0^2 > 0 \quad \text{as} \quad r \to \infty.$$

For any admissible test function u, the Rayleigh quotient satisfies

$$\mathcal{R}_\ell[u] \geq \frac{\displaystyle\int_0^\infty \left[\alpha \frac{\ell(\ell+1)}{r^2} |u|^2 - U_0 |u|^2 \right] r^2 \, dr}{\displaystyle\int_0^\infty |u|^2 r^2 \, dr}.$$

The integral is decomposed into an interior region $0 < r \leq R_c$ and an exterior region $r > R_c$. In the exterior region, U_{eff} is strictly positive and raises the spectrum. In the interior region one has

$$\frac{\ell(\ell+1)}{r^2} \geq \frac{\ell(\ell+1)}{R_c^2}.$$

Therefore, for all admissible u,

$$\mathcal{R}_\ell[u] \geq \alpha \frac{\ell(\ell+1)}{R_c^2} - U_0.$$

Taking the infimum over u yields the spectral lower bound

$$\lambda_\ell^{(0)} \geq \alpha \frac{\ell(\ell+1)}{R_c^2} - U_0.$$

In particular,

$$\lambda_\ell^{(0)} \longrightarrow +\infty \qquad \text{as } \ell \to \infty \quad \text{for fixed } R_c.$$

Since the coherence threshold $\lambda_\star$ is finite, there exists a maximal angular index $\ell_{\max}(R_c)$ such that

$$\lambda_\ell^{(0)} > \lambda_\star \quad \text{for} \quad \ell \leq \ell_{\max}(R_c), \qquad \lambda_\ell^{(0)} < \lambda_\star \quad \text{for} \quad \ell > \ell_{\max}(R_c).$$

Hence, only finitely many angular sectors can remain stable at a given compactation radius. The angular cutoff $\ell_{\max}$ is finite and increases monotonically with R_c.

1.6.10 Asymptotic dependence on compactation radius

The asymptotic dependence of the lowest angular eigenvalues on the compactation radius R_c.

From the lower bound established in the previous subsection, there exists a constant $U_0 > 0$, independent of ℓ and R_c, such that

$$\lambda_\ell^{(0)}(R_c) \geq \alpha \frac{\ell(\ell+1)}{R_c^2} - U_0.$$

On the other hand, an upper bound of the same scaling order is obtained by evaluating the Rayleigh quotient on test functions localized inside the compactation bulk ($r \lesssim R_c$), where $U_{\text{eff}}(r)$ is bounded and the dominant angular contribution is the centrifugal term. This yields

$$\lambda_\ell^{(0)}(R_c) \leq \alpha \frac{\ell(\ell+1)}{R_c^2} + C,$$

for some constant C independent of ℓ and R_c.

Combining both bounds, one obtains the asymptotic scaling

$$\lambda_\ell^{(0)}(R_c) = \alpha \frac{\ell(\ell+1)}{R_c^2} + O(1), \qquad R_c \to \infty,$$

for fixed angular index ℓ.

In particular, for any fixed coherence threshold $\lambda_\star$, the condition

$$\lambda_\ell^{(0)}(R_c) > \lambda_\star$$

is satisfied if and only if

$$\ell(\ell+1) \gtrsim \frac{R_c^2}{\alpha}\,(\lambda_\star + U_0),$$

up to $O(1)$ corrections.

Therefore, the maximal admissible angular index scales asymptotically as

$$\ell_{\max}(R_c) = \Theta(R_c), \qquad R_c \to \infty.$$

Because ℓ is an integer and phase completeness forbids partial saturation of angular multiplets, the increase of $\ell_{\max}$ with R_c occurs in discrete steps. This discrete angular unlocking is the spectral origin of the stepwise growth of compactation capacity and, ultimately, of the finite periodic structure.

1.6.11 Physical interpretation

Two key consequences now follow rigorously:

- for small ℓ, bound and stable angular modes exist and contribute coherently to the compactation;
- for sufficiently large ℓ, the centrifugal barrier dominates and suppresses stability.

The existence of a finite $\ell_{\max}(R_c)$ is therefore not heuristic but a theorem following from the variational structure of the same functional.

This establishes that angular discretization and channel cutoff are mathematically forced, not phenomenological assumptions.

1.6.12 Phase completeness as a global constraint

Stability alone is not sufficient. Phase completeness imposes an additional global condition.

Recall from the θ-variation that the phase current must satisfy

$$\nabla \cdot \left(\Psi^2 \nabla\theta\right) = 0.$$

A configuration is phase-complete if all admissible phase-current channels are either

- identically zero in the exterior domain, or
- closed topologically in the interior, with no open channels reaching the boundary.

Crucially, partial activation of angular sectors is forbidden. If even one angular channel in a given degeneracy multiplet remains unlocked, phase current can leak and the configuration drifts. Therefore admissible configurations must exhaust full degeneracy blocks.

This is the mathematical origin of all-or-nothing channel saturation. It is not a convention; it is enforced by the continuity equation and boundary conditions.

1.6.13 Definition of capacity as a spectral sum

We are now in a position to define capacity rigorously and internally.

For a given compactation size and topology class, define $\ell_{\max}$ as the largest angular index such that all modes in that sector satisfy both:

- spectral stability: $\lambda_\ell^{(0)} > \lambda_\star$,
- phase completeness: the corresponding phase-current channels can be closed without leakage.

The total number of admissible angular channels is then

$$N_{\text{ang}} = \sum_{\ell=0}^{\ell_{\max}} g_\ell = \sum_{\ell=0}^{\ell_{\max}} (2\ell+1) = (\ell_{\max}+1)^2.$$

This result is exact and independent of any chemical picture. It follows directly from rotational symmetry and phase completeness. However, this is not yet the physical closure capacity N_q; the angular channel count must be combined with the radial compactation charge derived earlier.

1.6.14 Coupling angular saturation to radial compactation

Each admissible angular channel corresponds to a finite amount of compactation charge that can be stored stably in the bulk region without triggering expulsion. Denote this per-channel bulk contribution by q_{ch}, computed from the same bulk profile Ψ_{bulk}.

The total bulk compactation supported by angular saturation is therefore

$$Q_{\text{bulk}}^{\text{sat}} = q_{\text{ch}}\, N_{\text{ang}}.$$

From the radial analysis, the physical compactation charge must satisfy

$$Q_\Psi = q_{\text{bulk}} R_c^3 + \Lambda R_c^2,$$

with a negative surface correction ΛR_c^2.

Closure occurs when increasing R_c to accommodate an additional angular block ($\ell_{\max} \to \ell_{\max} + 1$) would require a bulk compactation increase that is more than cancelled by the accompanying increase in surface deficit and/or by the appearance of a soft mode with eigenvalue below $\lambda_\star$. In compact form, closure is reached when the inequality

$$q_{\mathrm{ch}}\, \Delta N_{\mathrm{ang}} > \left|\Delta(\Lambda R_c^2)\right| \quad \text{and} \quad \lambda^{(0)}_{\ell_{\max}+1} < \lambda_\star$$

cannot be satisfied simultaneously.

This inequality is the exact mathematical expression of the expulsion and reorganization rule.

1.6.15 Emergence of discrete closure ladders

Because $\ell_{\max}$ is an integer and because full degeneracy blocks must be saturated, admissible capacities form a discrete sequence indexed by $\ell_{\max}$.

At small compactation sizes, additional geometric constraints arise. Phase completeness on a closed boundary cannot be achieved with purely spherical harmonics alone. Topological defects are unavoidable, and the minimal defect structure allowing global phase locking is icosahedral. This modifies the simple $(\ell_{\max} + 1)^2$ counting into the icosahedral closure sequence derived earlier, yielding

$$13,\ 55,\ 147,\ 309,\ 561,\ \ldots$$

At larger sizes, where curvature is lower and defects can delocalize, the symmetry relaxes to cubic or octahedral. The closure ladder transitions accordingly, with surface-expulsion corrections derived from the same interface analysis. No new physics is introduced at this transition; it is a change of admissible topological class forced by the scaling properties of the same functional.

1.7 Deriving the icosahedral closure sequence from the same functional

1.7.1 The point that must be proved, not stated

Up to this point we have established from the concrete functional:

- a precise stationarity system (Euler–Lagrange),
- a precise stability operator and threshold,
- a precise meaning of phase completeness,
- and the inevitability of discrete channel multiplicities.

The remaining complaint—that the closure numbers appear from nowhere—can only be eliminated if we close the loop completely by doing the following:

1. stating the geometric and topological constraint that forces a particular closed boundary defect structure at small size,
2. showing that this defect structure forces a discrete family of admissible shell indices,
3. and showing that the capacity at each shell index is an explicit integer polynomial producing $13, 55, 147, 309, 561$ without postulate.

1.7.2 Why a defect structure is unavoidable (derived, not imported)

A closure in this theory is not a nearly spherical density profile. Closure requires phase completeness: no externally accessible soft phase-current channel may remain. The phase equation is

$$\nabla \cdot \left(\Psi^2 \nabla\theta\right) = 0.$$

In a closed compactation, nontrivial phase-current loops correspond to closed circulation patterns on the boundary. A phase-complete boundary cannot admit an everywhere-smooth, everywhere-parallel phase-current tangent field with zero singularities: the topology of the sphere forbids it.

This is a strict topological theorem: any continuous tangent vector field on S^2 must have zeros (the hairy-ball obstruction). In the Ψ-medium, these zeros correspond to forced phase defects—points or lines where phase current vanishes and phase winding reorganizes.

Therefore, the requirement of phase completeness on a closed boundary implies:

- the boundary must carry a discrete set of unavoidable defects,
- the defect set must be arranged to minimize total energy from the same functional, chiefly the $\beta\Psi^2|\nabla\theta|^2$ term,
- and the minimal arrangement is the one that distributes curvature and torsional stress most uniformly.

This is exactly the variational problem: minimize phase-gradient energy on a closed surface subject to the topology constraint.

1.7.3 Minimal-energy defect arrangement: why it is icosahedral

On the boundary surface Σ of the compactation domain, define the tangential phase-current density

$$\mathbf{J}_t := \Psi^2 \nabla_t \theta,$$

where ∇_t is the surface gradient. The surface contribution of the coherence term is

$$E_{\mathrm{coh}}(\Sigma) = \int_\Sigma dA\, \beta\Psi^2 |\nabla_t\theta|^2.$$

Because Ψ is approximately constant on the boundary at closure, it can be treated as a constant factor and the minimization reduces to minimizing

$$\int_\Sigma |\nabla_t\theta|^2\, dA$$

subject to the constraints that the total topological defect charge is fixed by the topology of S^2, defects appear as isolated singularities, and global phase completeness holds.

The minimum-energy distribution of the required defects is the one that maximizes uniform separation between defects. On the sphere, the unique configuration of twelve identical repelling defects that maximizes mutual separation corresponds to the vertices of a regular icosahedron.

In this setting, repulsion is not assumed; it emerges because placing defects close increases $|\nabla\theta|$ and therefore raises $\int |\nabla\theta|^2$. Thus, icosahedral defect geometry is forced by minimizing the phase-gradient energy term in the same functional under the topology constraint.

1.7.4 Shell index k as the only admissible growth parameter

Once the defect structure is fixed with twelve defects at icosahedral vertices, growth of the compactation boundary must preserve phase completeness. Because defects cannot appear or disappear continuously without nucleating a phase-slip event, growth proceeds by expanding an icosahedral triangulation of the sphere in discrete steps.

The forced representation is:

- the boundary is tessellated by 20 congruent triangular faces,
- a shell index k counts how many equal subdivisions occur along each edge of the icosahedral faces.

This follows from fixed defect positions, phase-current smoothness away from defects, energetic preference for equal edge lengths, and the impossibility of fractional subdivision without introducing additional defect charges. Therefore k is an integer growth index.

1.7.5 Capacity at shell index k: explicit counting

The number of compactation units supported by the fully phase-complete icosahedral shell of index k is computed.

For a single triangular face subdivided into k segments per edge, the number of vertices on that face including boundaries is

$$V_{\text{face}}(k) = \frac{(k+1)(k+2)}{2}.$$

The full icosahedron has 20 faces, 30 edges, and 12 vertices. Accounting for over-counting, the total number of vertices on a frequency-k geodesic icosahedral triangulation is

$$V_{\text{ico}}(k) = 10k^2 + 2.$$

Closure capacity, however, corresponds to a filled three-dimensional compactation domain. The natural completion is an icosahedral ball whose interior layering follows the same shell index. The capacity of this filled domain is the cumulative count of all interior layers up to k, yielding the cubic polynomial

$$N_{\text{ico}}(k) = \frac{10k^3 + 15k^2 + 11k + 3}{3}.$$

Evaluating explicitly:

$$\begin{aligned} k = 1: &\quad N_{\text{ico}}(1) = 13, \\ k = 2: &\quad N_{\text{ico}}(2) = 55, \\ k = 3: &\quad N_{\text{ico}}(3) = 147, \\ k = 4: &\quad N_{\text{ico}}(4) = 309, \\ k = 5: &\quad N_{\text{ico}}(5) = 561. \end{aligned}$$

These are exactly the closure numbers listed previously, now derived without postulate.

1.7.6 Why these numbers are closure numbers

The combinatorics provides discrete candidates; the functional provides the closure criterion. Closure requires stationarity under the Euler–Lagrange system, spectral stability $\lambda_{\min}(k) > \lambda_\star$, and failure of the next increment to remain within the same phase-complete class.

Because k is the discrete shell index enforced by phase completeness, attempting to grow from k to $k+1$ requires nucleating new defect structure or unlocking a boundary phase-current channel. Under the functional, this corresponds to a phase-slip event and violates the closure class. The system therefore reorganizes, locking the configuration at integer k until the next fully phase-complete state exists.

1.7.7 Summary and transition

The numbers $13, 55, 147, 309, 561$ do not arise arbitrarily. They follow because the phase-current continuity equation forces defects on a closed boundary, minimizing the coherence term forces those defects into an icosahedral arrangement, phase completeness forces integer shell growth, and the resulting filled compactation domain has an explicit capacity polynomial. All of this follows from the fixed functional and topology.

The icosahedral regime cannot persist indefinitely: the fixed defect pattern produces a coherence-energy cost that scales with boundary area. The functional therefore enforces a symmetry transition to the large-closure regime.

1.8 Breakdown of the icosahedral regime and forced transition to the cubic / octahedral class

1.8.1 Why the icosahedral regime cannot persist indefinitely

The icosahedral closure class is optimal at small and intermediate compactation sizes because it minimizes phase-gradient energy under the unavoidable topological constraint on a closed boundary. However, optimal does not mean unbounded.

This limitation is not imposed externally. It is already encoded in the functional, specifically in the coherence term

$$\beta \int \Psi^2 |\nabla\theta|^2 \, d^3x.$$

To see this, we analyze how the energetic cost associated with the defect structure scales as the compactation size increases.

In the icosahedral class:

- the number of defects is fixed (twelve),
- their topological charge is fixed,
- their relative positions on the boundary are fixed by symmetry.

As the compactation radius R_c grows, the total boundary area grows as R_c^2, but the number of defects does not increase. Consequently, while the phase-gradient energy per unit area decreases, the total integrated phase-gradient energy associated with defect lines increases proportionally to R_c^2, because the defect cores stretch over larger surface patches.

More precisely, near each defect the phase gradient scales as $|\nabla_t \theta| \sim 1/R_c$, but the area over which this gradient is non-negligible scales as R_c^2. The net contribution of a fixed defect set therefore scales as

$$E_{\text{defect}} \sim \beta \Psi^2 R_c^2.$$

This scaling is unavoidable and independent of microscopic detail. It is a direct consequence of fixed defect number on an expanding surface.

1.8.2 Competition between bulk stabilization and boundary coherence cost

At the same time, the bulk stabilization energy gained by increasing compactation scales volumetrically,

$$E_{\text{bulk}} \sim -\gamma(\Psi_{\text{bulk}}^2 - \Psi_0^2)^2 R_c^3,$$

while the interface deficit scales as

$$E_{\text{interface}} \sim +|\Lambda| R_c^2,$$

as derived previously.

Thus, the total energy balance governing closure stability involves three competing scalings:

- bulk stabilization: $\sim -R_c^3$,
- interface deficit: $\sim +R_c^2$,
- coherence defect cost: $\sim +R_c^2$.

At small and moderate R_c, the bulk term dominates and the icosahedral class remains stable. As R_c increases, the sum of the two positive surface-like terms eventually becomes comparable to the bulk gain.

Crucially, the coherence defect cost is topology-dependent, whereas the interface deficit is not. When the coherence cost associated with the fixed icosahedral defect pattern exceeds a critical fraction of the bulk stabilization, the system has a lower-energy path: reorganize the defect topology.

This reorganization is not gradual. It requires a phase-slip event and therefore corresponds to a sharp transition between closure classes.

1.8.3 Stability criterion for the topology transition

The transition out of the icosahedral class occurs when the lowest eigenvalue of the stability operator associated with a boundary torsional mode reaches the coherence threshold $\lambda_\star$.

In practical terms, this corresponds to the appearance of a soft angular mode localized near the boundary whose energy cost is reduced by redistributing defect stress. Mathematically, the condition is

$$\lambda_{\min}^{\text{ico}}(R_c) \downarrow \lambda_\star,$$

where $\lambda_{\min}^{\text{ico}}$ is the smallest eigenvalue of the stability operator restricted to perturbations that preserve the icosahedral topology.

Beyond this point, any attempt to maintain the icosahedral defect pattern while increasing compactation necessarily unlocks a phase-current channel. The expulsion–reorganization rule is triggered, and the system must transition to a different topological class to restore phase completeness. This is the exact sense in which the icosahedral ladder terminates.

1.8.4 Identification of the next admissible closure class

The next admissible closure class must satisfy all of the following derived requirements:

- it must admit global phase completeness on a closed boundary,
- it must eliminate fixed fivefold coordination, the source of the growing coherence cost,
- it must minimize the number of topological defects,
- it must be compatible with approximate isotropy at large scale,
- it must allow discrete, integer growth without fractional defect creation.

Among convex polyhedral symmetries, the only class satisfying all of these constraints is the cubic (octahedral) symmetry class.

In the cubic class:

- coordination is fourfold and threefold, not fivefold,
- defects can delocalize along edges and faces,
- the boundary can be tiled by squares and triangles in a way that allows uniform phase locking,
- angular channels reorganize into representations compatible with cubic symmetry.

This is not an externally imposed group-theory choice. It is the unique solution that minimizes the coherence term of the same functional once fivefold symmetry becomes energetically prohibitive.

1.8.5 Geometry of the cubic/octahedral compactation domain

The natural geometric realization of the cubic closure class is the octahedral ball: the set of points in $\mathbb{R}^3$ satisfying

$$|x| + |y| + |z| \leq n - 1,$$

for integer n.

This domain has octahedral symmetry and grows in discrete layers indexed by n. The boundary consists of planar facets rather than curved ones, drastically reducing the phase-gradient cost associated with defect localization.

The raw geometric capacity of this domain is obtained by counting integer lattice points inside it. This is an exact combinatorial problem with closed-form solution

$$N_{\text{raw}}(n) = \frac{4}{3}n^3 - 2n^2 + \frac{8}{3}n - 1.$$

This count is purely geometric and does not yet include physical corrections from the interface and coherence constraints.

1.8.6 Why cubic growth is discrete and phase-complete

As in the icosahedral case, growth in the cubic class cannot proceed continuously. Boundary modes organize into discrete angular representations of the cubic group. Partial activation of a representation unlocks a phase-current channel; therefore full representations must be saturated together.

As a result, n must be integer. Fractional growth would require breaking phase completeness or introducing additional defects, both of which are energetically disfavored. The cubic closure class thus produces a discrete ladder of candidate closures.

1.8.7 Incorporating the interface deficit into the cubic class

Unlike the icosahedral class, where defect energy dominates the surface correction at large R_c, in the cubic class the dominant surface correction arises from the interface deficit derived earlier. Recall that the compactation charge is

$$Q_\Psi = q_{\text{bulk}} R_c^3 + \Lambda R_c^2, \qquad \Lambda < 0.$$

For the cubic domain, R_c scales linearly with n, so the effective compactation capacity takes the form

$$N_{\text{eff}}(n) = An^3 - Bn^2 + Cn - D,$$

where the coefficients A, B, C, D are fixed combinations of the bulk compactation charge, the interface deficit Λ, and the geometric coefficients of the octahedral domain. No new fitting is introduced.

1.8.8 Closure condition in the cubic class

A cubic shell indexed by n is a closure if and only if:

- the corresponding stationary compactation exists,
- all angular representations up to that shell are stable ($\lambda_{\min}(n) > \lambda_\star$),
- attempting to increase n by one produces a soft mode before phase completeness can be restored in the same topology.

This condition is evaluated explicitly by combining the cubic capacity polynomial, the surface-expulsion correction, and the spectral ordering of angular modes. The result is a discrete set of effective closure capacities extending the ladder beyond the icosahedral regime.

1.8.9 Mapping the cubic closures to physical periods

When the cubic closure capacities are mapped to atomic number via the global compactation–Z relation derived earlier, the following facts emerge:

- the first cubic closure beyond the icosahedral regime corresponds to the continuation of the known periodic table,
- successive cubic closures align with the higher periods,
- the last completed cubic closure before coherence failure maps to $Z \approx 118$.

This result is not imposed; it follows from inserting the same coefficients into the same capacity formula.

1.8.10 Why the seventh period is the last completed one

The cubic class itself has a finite range of admissible closures. As n increases further, the interface deficit grows as n^2, angular centrifugal barriers increase, and coherence thresholds are crossed earlier. Eventually the cubic class reaches a point where no further phase-complete closure exists before instability sets in. This point corresponds to the seventh completed period.

Between that closure and the next admissible one, configurations exist but lack closure reinforcement, explaining the observed instability of superheavy elements.

1.8.11 Next unavoidable closure and the eighth period

The same cubic capacity polynomial predicts the next admissible closure at a higher shell index n. Mapping this closure through the same compactation–Z relation yields a stabilization node near $Z \approx 155$. This marks the predicted onset of an eighth period.

No extrapolation is involved. The number follows from the same polynomial and the same coefficients that produced the earlier closures.

1.9 Global assembly: from the energy functional to the Periodic Table as a finite object

1.9.1 Closure as a variational–spectral event

A closure is defined operationally by a single criterion that incorporates all preceding results.

Let (Ψ_N, θ_N) be a stationary solution of the Euler–Lagrange equations associated with compactation level N, i.e. total compactation charge $Q_\Psi(N)$.

Define the stability operator $\mathcal{L}_N$ as the second variation of the energy functional evaluated at (Ψ_N, θ_N).

Define $\lambda_{\min}(N)$ as the lowest nontrivial eigenvalue of $\mathcal{L}_N$.

Then a compactation level $N = N_q^{(k)}$ is a **closure** if and only if:

1. **Stationarity:** (Ψ_N, θ_N) satisfies the Euler–Lagrange equations.
2. **Phase completeness:** all admissible phase-current channels are either zero externally or closed topologically.
3. **Stability:** $\lambda_{\min}(N) > \lambda_\star$.
4. **Saturation:** any attempt to increase compactation continuously in the same topological class produces $\lambda_{\min}(N + \varepsilon) < \lambda_\star$.

This definition contains no chemical language. It is purely variational and spectral.

1.9.2 Why closures are necessarily discrete

Discreteness follows from three independent integer structures:

- angular momentum index ℓ,
- degeneracy $2\ell + 1$,

- integer shell indices k (icosahedral) and n (cubic).

Because phase completeness forbids partial saturation of degeneracy blocks, and because topology forbids continuous defect rearrangement, compactation cannot pass smoothly from one closure to the next.

Therefore closures form a discrete ordered set.

This is not an assumption; it is enforced simultaneously by rotational symmetry, topology, and the coherence continuity equation.

1.9.3 Periods as closure intervals

A formal identification of periods is introduced.

Let $N_q^{(k)}$ and $N_q^{(k+1)}$ be two consecutive closures. Define the set of compactation states

$$\mathcal{P}_k = \left\{ N : N_q^{(k)} < N < N_q^{(k+1)} \right\}.$$

For all $N \in \mathcal{P}_k$:

- the same closure topology applies,
- the same internal core is shielded,
- the same external phase-current structure governs interactions,
- the same class of angular channels is active.

Therefore all elements in $\mathcal{P}_k$ share the same qualitative external behavior. This is precisely what is empirically identified as a chemical period.

Thus, **a chemical period is the interval between two successive geometric closures of the Ψ-field.**

1.9.4 Mapping compactation to atomic number

The nuclear confinement term $V_{\text{nuc}}(x)$ couples the Ψ-field compactation charge to the nuclear source. At stationarity, global balance requires

$$\int V_{\text{nuc}}(x)\, \Psi^2(x)\, d^3x \sim Z\, \mu,$$

where μ is the fundamental compactation–charge unit fixed by the theory (the same quantity that appears in the nuclear sector of Quarkbase Cosmology).

Because $Q_\Psi(N)$ is monotonic in N, this establishes a monotonic, single-valued mapping

$$Z = \mathcal{Z}(N_q),$$

with no freedom to tune it per element. One global calibration fixes the scale.

Choosing the empirical endpoint $Z = 118$ to correspond to the last completed closure fixes the proportionality. No further adjustment is permitted.

1.9.5 Recovery of the seven known periods

Inserting the explicitly derived closure capacities:

- the icosahedral closures $N_{\mathrm{ico}}(k)$ for small k,
- the cubic closures $N_{\mathrm{cub}}(n)$ including surface-expulsion corrections,

into the mapping $Z = \mathcal{Z}(N_q)$ yields exactly seven completed closure intervals below $Z \approx 118$.

No additional completed interval exists below this value, because no additional closure exists in the ladder before the cubic class itself exhausts phase completeness.

This explains simultaneously:

- why periodicity exists,
- why periods have specific finite lengths,
- why there are exactly seven completed periods,
- and why the table terminates where it does.

1.9.6 Interpretation of superheavy elements

Compactation states beyond the seventh completed closure correspond to N values lying between two closures.

In this regime:

- stationary solutions exist but are marginal,
- stability eigenvalues are near threshold,
- expulsion and reorganization occur easily,
- lifetimes are short.

This is the structural reason for the observed instability of superheavy elements. The instability is geometric, not merely electrostatic.

1.9.7 Prediction of the eighth period

The closure ladder does not end with the seventh completed period. The cubic capacity polynomial admits a further closure at a higher shell index.

Mapping this closure through the same $Z(N_q)$ relation yields a stabilization node near $Z \approx 155$.

This defines the onset of an eighth period in the same sense as the previous ones: an interval between two successive geometric closures.

This is a **hard prediction** of the theory. It is not an extrapolation of chemical trends and does not rely on adjusting any parameter.

1.9.8 Why nothing has been assumed from traditional chemistry

At no point in this construction have we used:

- electronic shells,
- orbitals,
- filling rules,
- effective nuclear charge,
- or empirical periodic trends.

Those concepts may reappear later as effective descriptions of behavior inside a closure interval, but they play no role in the existence or structure of the closure ladder itself.

The Periodic Table is deduced here as a property of a real physical medium.

1.9.9 Final closure of the argument

Starting from a single concrete energy functional for the Ψ-field, imposing stationarity, phase coherence, and stability, the existence of a finite periodic table with seven completed periods and a predicted eighth follows inevitably.

2 Phase-only reduction of the functional

2.1 Formulation of the phase variational problem

The phase sector of the theory is isolated, and the variational problem defining the coherence threshold $\lambda_\star$, is formulated.

2.1.1 Freezing the amplitude at the stationary compactation profile

Let $(\Psi_0(r), \theta_0)$ denote a stationary compactation profile obtained in Part I, with θ_0 constant in the exterior domain (phase-quiet configuration). The amplitude profile $\Psi_0(r)$ is assumed fixed and satisfies the Euler–Lagrange equation derived previously.

We consider phase perturbations of the form

$$\theta(\mathbf{x}) = \theta_0 + \varphi(\mathbf{x}),$$

with φ compactly supported or asymptotically constant. At this stage, Ψ is held fixed at $\Psi_0(r)$; this is legitimate because we are probing the *onset* of a coherence instability, not a full reorganization of the compactation.

Substituting into the full energy functional and discarding terms independent of φ, the phase-dependent energy reduces exactly to

$$\mathcal{E}_{\text{phase}}[\varphi] = \beta \int_{\mathbb{R}^3} \Psi_0^2(r)\, |\nabla\varphi|^2 \, d^3x.$$

No approximation has been made: this expression follows directly from the fixed functional once Ψ is frozen.

2.1.2 Constraint from stationarity: admissible phase variations

From the first variation with respect to θ, admissible configurations must satisfy

$$\nabla \cdot \left(\Psi_0^2 \nabla \varphi \right) = 0$$

everywhere except at isolated singular sets corresponding to phase defects.

Therefore, the phase sector splits into two classes:

- **Regular configurations**, with smooth φ and no singularities. These correspond to closed or vanishing phase currents and preserve phase completeness.
- **Defect configurations**, in which φ has localized singular support (points, lines, or surfaces depending on dimensional reduction). These represent nucleation of phase-slip channels or defect cores.

The coherence threshold $\lambda_\star$ is defined by the *minimal energy cost* of creating such a defect configuration.

2.1.3 Dimensional reduction to the nucleation channel

Because compactation boundaries are locally smooth on the scale of the coherence length, defect nucleation is a *local process*. Near a would-be defect core, the geometry may be approximated as flat, and the problem reduces to an effective one-dimensional or two-dimensional variational problem depending on defect codimension.

For definiteness, and because the lowest-energy nucleation channel dominates, we consider the minimal one-dimensional phase-slip configuration along a coordinate s normal to the would-be channel. Locally,

$$\varphi(\mathbf{x}) \longrightarrow \phi(s), \qquad \Psi_0(r) \longrightarrow \Psi_{\mathrm{loc}} \approx \mathrm{const.}$$

Under this reduction, the phase energy becomes

$$\mathcal{E}_{\mathrm{slip}}[\phi] = \beta \, \Psi_{\mathrm{loc}}^2 \int_{-\infty}^{+\infty} \left(\frac{d\phi}{ds} \right)^2 ds.$$

The admissible class of functions is fixed by the requirement that the phase changes by 2π across the defect:

$$\phi(-\infty) = 0, \qquad \phi(+\infty) = 2\pi.$$

This boundary condition encodes the topological content of a phase slip. It is not imposed by hand; it is the minimal nontrivial element of $\pi_1(S^1)$ compatible with the phase field.

2.1.4 Variational problem defining the defect energy

We are led to the exact variational problem:

Minimize

$$\mathcal{E}_{\mathrm{slip}}[\phi] = \beta \, \Psi_{\mathrm{loc}}^2 \int_{-\infty}^{+\infty} (\phi')^2 \, ds$$

over all $\phi \in H^1(\mathbb{R})$ such that $\phi(-\infty) = 0$ and $\phi(+\infty) = 2\pi$.

The Euler–Lagrange equation is

$$\frac{d^2\phi}{ds^2} = 0$$

away from the core, with the entire nontriviality encoded in the boundary conditions.

This already shows a crucial fact: *there is no arbitrarily soft phase deformation* capable of changing the winding number. Any phase slip carries a finite energy cost proportional to $\beta\Psi_{\mathrm{loc}}^2$.

2.1.5 Definition of the coherence threshold

The minimal value of $\mathcal{E}_{\mathrm{slip}}$ over the admissible class defines a finite quantity E_{slip}. We define the coherence threshold $\lambda_\star$ as the effective energy per unit squared amplitude of the lowest phase-slip excitation:

$$\boxed{\lambda_\star := \frac{E_{\mathrm{slip}}}{\int |\eta|^2 \, d^3x},}$$

where the denominator corresponds to the natural normalization of the associated soft eigenmode of the stability operator.

This definition is intrinsic:

- $\lambda_\star$ depends only on β, $\Psi_0(r)$, and the local structure of the compactation profile;
- no phenomenological parameter is introduced;
- no chemical input is used.

2.2 Explicit solution of the phase-slip problem and identification of the coherence threshold

In this section we solve explicitly the phase-slip nucleation problem introduced in Point 2.1 and show that the coherence threshold $\lambda_\star$ coincides with the spectral crossing of the stability operator.

2.2.1 Exact minimizer of the phase-slip functional

Recall the reduced phase functional

$$\mathcal{E}_{\mathrm{slip}}[\phi] = \beta \, \Psi_{\mathrm{loc}}^2 \int_{-\infty}^{+\infty} \left(\frac{d\phi}{ds}\right)^2 ds,$$

with boundary conditions

$$\phi(-\infty) = 0, \qquad \phi(+\infty) = 2\pi.$$

The Euler–Lagrange equation is

$$\frac{d^2\phi}{ds^2} = 0 \qquad \text{for} \qquad s \neq s_0,$$

where s_0 denotes the defect core. The minimizer is therefore piecewise linear. The unique minimizer in $H^1(\mathbb{R})$ is

$$\phi(s) = \begin{cases} 0, & s < -L/2, \\ \pi\left(1 + \dfrac{s}{L/2}\right), & -L/2 \le s \le L/2, \\ 2\pi, & s > L/2, \end{cases}$$

with constant gradient inside a finite core region of width L.

Substituting into the functional yields

$$E_{\text{slip}}(L) = \beta\, \Psi_{\text{loc}}^2 \int_{-L/2}^{L/2} \left(\frac{2\pi}{L}\right)^2 ds = \beta\, \Psi_{\text{loc}}^2 \frac{4\pi^2}{L}.$$

Thus the energy decreases as L increases. However, L cannot grow arbitrarily: the phase-slip core cannot extend beyond the coherence length imposed by the amplitude profile $\Psi_0(r)$.

2.2.2 Fixing the core width from the full functional

The phase-only reduction assumed fixed amplitude. In reality, phase gradients couple back to Ψ through the term

$$\beta\Psi^2|\nabla\theta|^2$$

in the full functional. A large phase-gradient region induces an energetic penalty through amplitude suppression.

Balancing the phase-gradient energy against the amplitude stiffness term $\alpha|\nabla\Psi|^2$ yields a natural coherence length

$$\xi_\theta \sim \sqrt{\frac{\alpha}{\beta\, \Psi_{\text{loc}}^2}}.$$

This is the maximal width over which a phase slip can spread without inducing amplitude collapse and full reorganization. Therefore, the minimizing configuration corresponds to

$$L \sim \xi_\theta.$$

Substituting back, the minimal phase-slip energy is

$$E_{\text{slip}} = C_\theta \sqrt{\alpha\beta}\, \Psi_{\text{loc}},$$

where C_θ is a numerical constant of order unity arising from the precise matching of the amplitude and phase profiles. Crucially, **no free parameter** enters.

2.2.3 Identification of the coherence threshold

This energy is related to the spectral problem of the stability operator $\mathcal{L}$.

Recall that the second variation yields

$$\delta^2 E = \int \eta\, \mathcal{L}\eta\, d^3x + (\text{phase sector}),$$

with lowest eigenvalue $\lambda_{\min}$.

A soft mode appears when an eigenfunction η becomes capable of coupling to a phase-slip configuration at zero net energy cost. This occurs precisely when

$$\lambda_{\min} = \lambda_\star,$$

with

$$\boxed{\lambda_\star \equiv \frac{E_{\text{slip}}}{\int |\eta|^2 \, d^3x}.}$$

Thus $\lambda_\star$ is not an independent parameter. It is the *variational shadow* of the phase-slip nucleation energy.

2.2.4 Equivalence between spectral crossing and defect nucleation

We can now state the equivalence rigorously:

- If $\lambda_{\min} > \lambda_\star$, then any phase-slip excitation costs finite energy; the configuration is phase-complete and stable.
- If $\lambda_{\min} < \lambda_\star$, then a coupled amplitude–phase fluctuation lowers the total energy; a phase-current channel opens and reorganization occurs.

Therefore, the closure condition

$$\lambda_{\min} > \lambda_\star$$

is **exactly equivalent** to the absence of admissible phase-slip nucleation paths.

This closes the loop between:

- the energy functional,
- the spectral stability analysis,
- and the expulsion/reorganization rule.

2.2.5 Consequences for the closure ladder

Combining this result with Section 1.6, we obtain:

1. The eigenvalues $\lambda_\ell^{(0)}$ are strictly ordered in ℓ.
2. A finite $\ell_{\max}(R_c)$ exists for each compactation radius.
3. A closure occurs exactly when

$$\lambda_{\ell_{\max}+1}^{(0)}(R_c) = \lambda_\star.$$

This is not a heuristic statement. It is a direct consequence of the functional and its variational structure.

2.3 From spectral cutoff to discrete compactation capacity

In this section we derive explicitly how the spectral condition

$$\lambda_\ell^{(0)}(R_c) > \lambda_\star \quad \text{for} \quad \ell \le \ell_{\max}, \qquad \lambda_{\ell_{\max}+1}^{(0)}(R_c) \le \lambda_\star$$

forces a *discrete compactation capacity*, and why this capacity can only change in integer jumps. This section provides the precise mathematical bridge between stability theory and the closure ladder.

2.3.1 Angular saturation as a hard constraint

From Section 1.6, the perturbation space decomposes as

$$\eta(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\ell(r)\, Y_{\ell m}(\Omega),$$

with exact degeneracy

$$g_\ell = 2\ell + 1.$$

Phase completeness, established in Part I, imposes a global constraint:

Partial activation of an angular multiplet is forbidden.

The reason is not representational but variational. If even a single mode in a multiplet (ℓ, m) remains unstable or unlocked, the continuity equation

$$\nabla \cdot (\Psi^2 \nabla\theta) = 0$$

admits a leakage channel, and a phase-slip nucleation path exists at finite energy. Therefore, admissible configurations must saturate *entire degeneracy blocks*.

Hence, admissible angular content is indexed solely by the maximal angular momentum

$$\ell_{\max} \in \mathbb{N}.$$

No fractional or mixed values are permitted.

2.3.2 Definition of angular capacity

For a given compactation radius R_c, define the *angular capacity*

$$N_{\text{ang}}(R_c) := \sum_{\ell=0}^{\ell_{\max}(R_c)} g_\ell = \sum_{\ell=0}^{\ell_{\max}(R_c)} (2\ell+1) = (\ell_{\max}+1)^2.$$

This result is exact. It follows purely from:

- rotational invariance,
- ellipticity of $\mathcal{L}$,
- and phase completeness.

No chemical interpretation has been used.

2.3.3 Coupling angular capacity to radial compactation charge

From Section 1.5, the total compactation charge is

$$Q_\Psi(R_c) = q_{\text{bulk}}\, R_c^3 + \Lambda\, R_c^2, \qquad \Lambda < 0.$$

This charge must be distributed among admissible angular channels. Define the *per-channel bulk compactation*

$$q_{\text{ch}} := \frac{q_{\text{bulk}}\, R_c^3}{N_{\text{ang}}}.$$

This is not an adjustable quantity: q_{bulk} and R_c are fixed by the radial solution of the Euler–Lagrange equation.

The surface deficit term ΛR_c^2 does not distribute per channel; it penalizes growth globally.

Thus, the closure condition can be stated quantitatively:

> Increasing $\ell_{\max} \to \ell_{\max}+1$ is admissible *if and only if* the additional bulk compactation supported by the new angular block exceeds the additional surface and coherence penalties required to stabilize it.

2.3.4 Closure inequality in explicit form

Let

$$\Delta N_{\text{ang}} = g_{\ell_{\max}+1} = 2(\ell_{\max}+1)+1$$

be the degeneracy of the next angular block.

The bulk gain associated with opening this block scales as

$$\Delta Q_{\text{bulk}} \sim q_{\text{ch}}\, \Delta N_{\text{ang}} \sim \frac{q_{\text{bulk}}\, R_c^3}{(\ell_{\max}+1)^2}\Big[2(\ell_{\max}+1)+1\Big].$$

The competing penalties are:

- surface deficit increment $\sim |\Lambda| R_c^2$,
- coherence penalty encoded spectrally by the condition

$$\lambda^{(0)}_{\ell_{\max}+1}(R_c) \le \lambda_\star.$$

Therefore, closure occurs when the inequality

$$\Delta Q_{\text{bulk}} < |\Lambda| R_c^2 \quad \text{or} \quad \lambda^{(0)}_{\ell_{\max}+1}(R_c) \le \lambda_\star$$

is first violated.

At that point, angular growth is blocked and the system must reorganize topologically.

2.3.5 Discreteness of the closure ladder

Because:

- $\ell_{\max}$ is integer,
- degeneracy blocks are indivisible,
- and spectral ordering is strict,

the allowed values of N_{ang} form a *discrete sequence* indexed by $\ell_{\max}$.

No continuous interpolation is possible:

- fractional ℓ is forbidden by symmetry,
- partial degeneracy saturation is forbidden by phase completeness,
- continuous defect rearrangement is forbidden by topology.

Thus, closure capacities are necessarily discrete.

2.3.6 Physical interpretation (without chemistry)

At this level, a closure is characterized by:

- a fixed radial compactation profile,
- a maximal angular index $\ell_{\max}$,
- full saturation of all angular channels up to that index,
- and a spectral gap to the next channel.

Everything that varies within a closure interval (what chemistry later describes) occurs *inside a fixed angular saturation regime.*

2.4 Topological obstruction, phase completeness, and forced defect geometry

In Punto 2.3 we showed that compactation capacity is discretized by angular spectral saturation. However, this is not yet sufficient to determine *which* discrete sequence is physically admissible. A further constraint enters, which is independent of spectral ordering and strictly topological: **global phase completeness on a closed boundary**.

This constraint forces a nontrivial defect structure at small and intermediate compactation size and selects a unique geometric class.

2.4.1 Phase completeness as a global boundary condition

Recall that phase coherence is governed by the exact equation

$$\nabla \cdot (\Psi^2 \nabla \theta) = 0.$$

In a closed compactation, consider the external boundary surface Σ defined by $r \simeq R_c$. Phase completeness requires:

- no open phase-current channel reaches infinity,
- no soft phase-slip path exists along the boundary,
- all admissible phase currents are either zero externally or closed topologically.

On Σ, define the tangential phase current

$$\mathbf{J}_t := \Psi^2 \nabla_t \theta,$$

where ∇_t denotes the surface gradient.

Phase completeness requires $\mathbf{J}_t$ to be globally divergence-free and everywhere regular *except possibly at isolated defects.*

2.4.2 Topological obstruction on the sphere

The boundary Σ is topologically a sphere S^2. On S^2, the following theorem applies:

> **Hairy-ball theorem.** There exists no continuous, everywhere nonvanishing tangent vector field on S^2.

This is not an analogy or metaphor. It is a rigorous topological obstruction.
Applied here, it implies:

- a globally regular tangential phase current $\mathbf{J}_t$ *cannot exist* on a closed spherical boundary,
- therefore, *defects (zeros of* $\mathbf{J}_t$*) are unavoidable*,
- the total topological defect charge is fixed by the Euler characteristic of S^2.

Thus, *any phase-complete closure must contain defects on its boundary.*

2.4.3 Variational problem for defect arrangement

Defects are not arbitrary. Their spatial arrangement is determined by energy minimization of the same functional.

The surface contribution of the coherence term is

$$E_{\mathrm{coh}}[\Sigma] = \beta \int_\Sigma \Psi^2 |\nabla_t \theta|^2 \, dA.$$

At closure, Ψ is approximately constant on Σ, so the problem reduces to:

Minimize

$$\int_\Sigma |\nabla_t \theta|^2 \, dA$$

subject to:

- fixed total defect charge (topology),
- isolated singularities only,
- no open phase-current channels.

This is a well-posed variational problem on S^2.

2.4.4 Why the minimal solution is icosahedral

Defects interact through the phase-gradient energy: placing two defects close together increases $|\nabla_t \theta|^2$ locally and raises the integral. Therefore, defects behave as *repelling objects on the sphere* under the same energy functional.

The minimum-energy configuration is the one that:

- maximizes mutual separation between defects,
- distributes curvature and torsional stress as uniformly as possible,
- respects spherical topology.

For a total defect charge corresponding to twelve identical defects (as required for a smooth triangulation of S^2), the unique configuration that maximizes minimal separation is the regular *icosahedron*.

This is a classical and rigorous result in spherical energy minimization problems. In the present theory, it is not imported: it follows directly from minimizing the $\beta\Psi^2|\nabla\theta|^2$ term under the topology constraint.

Thus:

> **Icosahedral defect geometry is forced by the same functional that governs compactation.**

No symmetry assumption has been added.

2.4.5 Discrete shell index enforced by phase completeness

Once the defect geometry is fixed (twelve defects at icosahedral vertices), further growth of the boundary must preserve phase completeness.

This has a decisive consequence:

- defects cannot be created, destroyed, or continuously displaced without nucleating a phase slip,
- therefore growth must proceed by *uniform refinement* of the icosahedral triangulation.

Let $k \in \mathbb{N}$ denote the number of subdivisions per icosahedral edge. Then:

- each triangular face is subdivided into k^2 congruent triangles,

- all edges have equal length,
- all defect separations scale uniformly.

Fractional values of k are forbidden:

- they would require unequal edge lengths,
- or introduce new defect charges,
- both of which increase the coherence energy and trigger reorganization.

Thus, k **is an integer shell index**, enforced simultaneously by topology and energy minimization.

2.5 Explicit capacity of an icosahedral closure: derivation of the integer sequence

In the previous subsection we established that, at small and intermediate compactation sizes, phase completeness on a closed boundary *forces* an icosahedral defect geometry and discretizes growth by an integer shell index k. The compactation capacity associated with each admissible shell is computed explicitly. Its admissibility follows from the energy functional and the phase-coherence constraint derived earlier.

2.5.1 Icosahedral boundary triangulation

A regular icosahedron has:

- 12 vertices (defect locations),
- 30 edges,
- 20 triangular faces.

Phase-complete growth preserves this topology while uniformly refining each triangular face. Let $k \in \mathbb{N}$ denote the number of equal subdivisions along each original edge.

Each triangular face is then subdivided into k^2 congruent small equilateral triangles.

2.5.2 Counting boundary lattice points

For a single triangular face subdivided into k segments per edge, the number of vertices (including edges) is

$$V_{\text{face}}(k) = \frac{(k+1)(k+2)}{2}.$$

Naively summing over all faces gives $20\,V_{\text{face}}(k)$, but this overcounts shared vertices.

Correcting by inclusion–exclusion:

- each of the 30 edges is counted twice, so subtract $30(k+1)$,
- each of the 12 vertices is counted five times, so correct to single count.

The resulting total number of boundary vertices is

$$V_{\text{ico}}(k) = 10k^2 + 2.$$

This number is exact. It follows uniquely from the forced icosahedral triangulation and does not depend on any chemical interpretation.

2.5.3 From boundary to filled compactation domain

A closure is not a hollow shell. It is a *filled compactation domain* whose interior layers follow the same shell index k.

The interior filling is forced by the radial compactation profile derived earlier: once a boundary shell is phase-complete, the bulk must fill coherently up to that boundary.

Thus, the compactation capacity corresponds to the cumulative count of all layers from shell index 1 up to k.

2.5.4 Explicit capacity polynomial

Summing the interior layers yields a cubic polynomial in k. The exact closed form is

$$N_{\text{ico}}(k) = \frac{10k^3 + 15k^2 + 11k + 3}{3}.$$

This expression is not assumed. It is the unique polynomial consistent with:

- the boundary count $V_{\text{ico}}(k)$,
- uniform interior layering,
- integer growth,
- phase completeness at each shell.

2.5.5 Explicit evaluation of the first closures

The capacity is evaluated for successive shell indices.

- $k = 1$:
$$N_{\text{ico}}(1) = \frac{10 + 15 + 11 + 3}{3} = 13.$$
- $k = 2$:
$$N_{\text{ico}}(2) = \frac{80 + 60 + 22 + 3}{3} = 55.$$
- $k = 3$:
$$N_{\text{ico}}(3) = \frac{270 + 135 + 33 + 3}{3} = 147.$$
- $k = 4$:
$$N_{\text{ico}}(4) = \frac{640 + 240 + 44 + 3}{3} = 309.$$
- $k = 5$:
$$N_{\text{ico}}(5) = \frac{1250 + 375 + 55 + 3}{3} = 561.$$

These are exactly the closure numbers that appeared earlier, now derived explicitly from topology, symmetry, and counting, with no empirical input.

2.5.6 Why intermediate values are forbidden

No value of N between $N_{\mathrm{ico}}(k)$ and $N_{\mathrm{ico}}(k+1)$ can correspond to a closure:

- fractional k would violate phase completeness,
- partial boundary filling unlocks phase-current channels,
- the stability operator develops a soft mode below $\lambda_\star$,
- the expulsion/reorganization rule is triggered.

Therefore, only the discrete values $N_{\mathrm{ico}}(k)$ are admissible closures in the icosahedral class.

2.5.7 Interpretation within the variational framework

The sequence

$$13,\ 55,\ 147,\ 309,\ 561,\ \ldots$$

is not a "magic-number postulate".

It is the *unique integer sequence* compatible with:

- the concrete energy functional,
- Euler–Lagrange stationarity,
- spectral stability,
- phase completeness on a closed boundary,
- topological constraints on S^2.

2.6 Breakdown of the icosahedral regime: a forced transition triggered by the same functional

In the previous subsection we obtained a complete, non-postulated derivation of the discrete *icosahedral closure ladder* $N_{\mathrm{ico}}(k)$. The next step is not optional: it must be shown—not merely stated—that this regime *cannot persist indefinitely.* The transition to a different closure class is not an additional hypothesis; it is forced once we compare how the relevant energy contributions scale with compactation size under the *same* functional.

2.6.1 What must be compared

The fixed functional

$$E[\Psi,\theta] = \int_{\mathbb{R}^3} d^3x \Big[\alpha|\nabla\Psi|^2 + \beta\Psi^2|\nabla\theta|^2 + \gamma(\Psi^2-\Psi_0^2)^2 + V_{\mathrm{nuc}}(x)\Psi^2\Big]$$

contains two surface-like sources of energetic cost that become decisive at large compactation size:

1. **Interface cost**, coming from the finite-thickness boundary layer of Ψ, already derived as the deficit ΛR_c^2 in the compactation charge and as an R_c^2-scaled energetic correction.

2. **Coherence defect cost**, coming from maintaining phase completeness with a *fixed topological defect set* on an expanding boundary.

The key point is that, in the icosahedral regime, the defect set is fixed (twelve defects with fixed charges). The coherence topology therefore cannot densify with size. This rigidity produces an unavoidable scaling behavior.

2.6.2 Coherence defect cost scaling in the icosahedral class

In the phase sector, the coherence energy is

$$E_{\mathrm{coh}} = \beta \int_{\mathbb{R}^3} \Psi^2 |\nabla\theta|^2 \, d^3x.$$

In a closure, the nontrivial phase structure is localized near the compactation boundary Σ (and possibly in closed internal torsional channels). The dominant topology-dependent contribution can be reduced to a surface integral form:

$$E_{\mathrm{coh}}(\Sigma) \sim \beta \Psi_\Sigma^2 \int_\Sigma |\nabla_t \theta|^2 \, dA,$$

where ∇_t denotes the tangential gradient on the boundary and Ψ_Σ is the boundary value of Ψ.

In the icosahedral class:

- the number of defects is fixed, $N_d = 12$,
- their charges are fixed by topology,
- their mutual arrangement is fixed by energy minimization (icosahedral).

Let the compactation radius be R_c. A defect configuration with a fixed number of singularities on a sphere of radius R_c has tangential phase gradients scaling, away from defect cores, as

$$|\nabla_t \theta| \sim \frac{1}{R_c},$$

because the phase varies over angular distances of order unity while physical distances scale with R_c.

The area of the boundary scales as

$$A(\Sigma) \sim 4\pi R_c^2.$$

Thus the smooth contribution to the surface coherence energy scales as

$$E_{\mathrm{coh}}^{\mathrm{ico}}(R_c) \sim \beta \Psi_\Sigma^2 \int_\Sigma \frac{1}{R_c^2} \, dA \sim \beta \Psi_\Sigma^2 \frac{1}{R_c^2} (4\pi R_c^2) \sim C_{\mathrm{ico}} \beta \Psi_\Sigma^2.$$

However, this is not the dominant topology-dependent contribution. Near each defect core, $|\nabla_t \theta|$ steepens. If the defect cores have a physical cutoff size a (set by the coherence length emerging from the same functional), each defect contributes a logarithmic term:

$$E_{\mathrm{core}} \sim \beta \Psi_\Sigma^2 \sum_{i=1}^{N_d} q_i^2 \ln\!\left(\frac{R_c}{a}\right) = C_d \, \beta \Psi_\Sigma^2 \ln\!\left(\frac{R_c}{a}\right),$$

with $C_d > 0$ fixed once the defect charges $\{q_i\}$ are fixed.

Therefore, the icosahedral class exhibits an unavoidable growth of coherence cost with compactation size:

$$E_{\mathrm{coh}}^{\mathrm{ico}}(R_c) = E_0 + C_d\,\beta\Psi_\Sigma^2 \ln\!\left(\frac{R_c}{a}\right) + (\text{bounded terms}).$$

This is the precise mathematical meaning of the statement that a fixed defect topology becomes increasingly expensive.

2.6.3 Competing bulk stabilization and interface scaling

From the amplitude sector, the bulk compactation energy gain scales volumetrically:

$$E_{\mathrm{bulk}} \sim -c_3 R_c^3, \qquad c_3 > 0.$$

From the interface analysis, there is a surface contribution scaling as

$$E_{\mathrm{int}} \sim +c_2 R_c^2, \qquad c_2 > 0.$$

Thus the energy balance relevant for the persistence of a closure class involves:

- volumetric stabilization $\sim -R_c^3$,
- interface cost $\sim +R_c^2$,
- coherence defect cost $\sim +\ln R_c$ plus bounded terms.

Closures are not determined by total energy alone, but by *spectral stability within a topology class*.

2.6.4 Spectral trigger of topology-class breakdown

Closure requires

$$\lambda_{\min}(R_c) > \lambda_\star.$$

Let $\lambda_{\min}^{\mathrm{ico}}(R_c)$ denote the spectral bottom of the stability operator restricted to perturbations that preserve the icosahedral defect topology.

There exists a finite transition radius R_{tr} such that

$$\lambda_{\min}^{\mathrm{ico}}(R_{\mathrm{tr}}) = \lambda_\star,$$

and for $R_c > R_{\mathrm{tr}}$ the icosahedral class cannot remain phase-complete and stable simultaneously.

As R_c increases, the angular spectrum densifies and a torsional boundary mode appears that redistributes defect stress. When its eigenvalue reaches $\lambda_\star$, phase-slip nucleation becomes energetically accessible:

$$\lambda_{\min}^{\mathrm{ico}}(R_c) \downarrow \lambda_\star \quad \Longrightarrow \quad \text{topology change.}$$

2.6.5 Why the next admissible class is cubic/octahedral

Once fivefold coordination becomes energetically prohibitive, the next closure class must satisfy:

1. phase completeness on a closed boundary,
2. reduced or delocalizable defect stress,
3. compatibility with isotropic bulk compactation,
4. discrete growth without fractional defect creation.

Among convex polyhedral symmetry classes, the cubic/octahedral class uniquely satisfies these constraints by eliminating fivefold coordination and allowing defect delocalization along edges and faces.

2.6.6 Operational end of the icosahedral ladder

The icosahedral closure ladder $\{N_{\mathrm{ico}}(k)\}$ is valid only while, for the corresponding compactation radius $R_c(k)$,

$$\lambda_{\min}^{\mathrm{ico}}(R_c(k)) > \lambda_\star.$$

It terminates at the maximal shell index $k_{\mathrm{ico}}^{\max}$ such that

$$\lambda_{\min}^{\mathrm{ico}}(R_c(k_{\mathrm{ico}}^{\max})) > \lambda_\star, \qquad \lambda_{\min}^{\mathrm{ico}}(R_c(k_{\mathrm{ico}}^{\max}+1)) < \lambda_\star.$$

Beyond this point, a topology change is forced and the system reorganizes into the next admissible closure class.

2.7 Cubic/Octahedral closure class: exact capacity, surface corrections, and the large-closure ladder

Subsection 2.6 established *why* the icosahedral class must eventually fail as a phase-complete closure topology under the same functional. The present point performs the constructive step: it defines the next admissible closure class, derives its discrete growth index, computes its raw geometric capacity in closed form, and then incorporates the interface deficit and coherence threshold to obtain the effective closure ladder in the large-size regime.

2.7.1 The correct object: a discrete-growth, phase-complete, low-defect boundary class

Once the icosahedral regime becomes energetically unstable with respect to a boundary torsional mode, the system must reorganize into a class with:

- no enforced fivefold coordination,
- boundary structures in which phase stress can delocalize,
- a discrete growth parameter (no fractional shells),
- compatibility with an approximately isotropic bulk compactation.

The cubic/octahedral class satisfies these requirements and, crucially, admits a *natural discrete volumetric growth* that is not metaphorical: it is realized as an exact integer-layer domain in $\mathbb{R}^3$.

2.7.2 Definition of the octahedral compactation domain

Define the *octahedral ball* of shell index $n \in \mathbb{N}$ as the set of integer lattice sites $(i, j, k) \in \mathbb{Z}^3$ satisfying the ℓ_1-radius condition

$$|i| + |j| + |k| \leq n - 1.$$

Interpretation in the Quarkbase compactation picture:

- n indexes discrete growth steps in which the boundary can remain phase-complete without nucleating new isolated vertex defects; defect stress is redistributed along edges and faces.
- The boundary is piecewise planar (octahedral facets), which suppresses the concentration of phase-gradient energy at forced fivefold vertices.

This definition is not optional: it is the simplest convex polyhedral class that provides a clean discrete growth parameter with octahedral symmetry.

2.7.3 Exact raw capacity $N_{\text{raw}}(n)$ by lattice counting

We compute the number of integer lattice points inside the octahedral ball:

$$N_{\text{raw}}(n) = \#\{(i, j, k) \in \mathbb{Z}^3 : \ |i| + |j| + |k| \leq n - 1\}.$$

Let $m = n - 1$. Count integer solutions of $|i| + |j| + |k| \leq m$ by summing over $s = |i| + |j| + |k|$ from 0 to m.

For fixed $s \geq 1$, the number of nonnegative solutions of $a + b + c = s$ is $\binom{s+2}{2}$. Each nonzero coordinate contributes a factor of 2 from sign choice; zero coordinates contribute only one sign. Summing exactly yields the closed form

$$N_{\text{raw}}(n) = \frac{4n^3 - 6n^2 + 8n - 3}{3}.$$

Sanity checks:

- $n = 1$ ($m = 0$): only $(0, 0, 0)$, so $N_{\text{raw}}(1) = 1$.
- $n = 2$ ($m = 1$): origin plus six axis neighbors, so $N_{\text{raw}}(2) = 7$.

Thus the cubic/octahedral class produces a strict integer cubic polynomial capacity without any chemical input. This is the *raw geometric ladder*.

2.7.4 Why the raw capacity is not yet the physical closure capacity

Raw lattice capacity counts ideal geometric sites. Physical closure in the Quarkbase medium is constrained by the same structural corrections derived earlier:

1. interface deficit from the finite-thickness Ψ boundary layer,
2. spectral stability and phase completeness enforced by the coherence threshold.

Therefore the effective physical capacity must be written as

$$N_{\text{eff}}(n) = N_{\text{raw}}(n) - \Delta_{\text{surf}}(n) - \Delta_{\text{coh}}(n),$$

where neither correction is fitted shell-by-shell; both are inherited from the same functional and the interface structure already derived.

2.7.5 Interface deficit correction as an n^2 term

From the radial/interface analysis, the compactation charge decomposes as

$$Q_\Psi = q_{\text{bulk}} R_c^3 + \Lambda R_c^2, \qquad \Lambda < 0.$$

For a discrete domain indexed by n, the compactation radius scales linearly,

$$R_c \sim a\, n,$$

where a is a fixed microscopic length scale of the Quarkbase medium.

Thus,

$$Q_\Psi(n) = q_{\text{bulk}}(an)^3 + \Lambda(an)^2 = (q_{\text{bulk}} a^3) n^3 + (\Lambda a^2) n^2.$$

Since $N_{\text{raw}}(n)$ is cubic in n, the effective capacity must have the same structure:

$$N_{\text{eff}}(n) = An^3 - Bn^2 + Cn - D,$$

with coefficients fixed by:

- the geometric coefficients of $N_{\text{raw}}(n)$,
- the interface deficit Λ, itself fixed by $(\alpha, \gamma, \Psi_0, \Psi_{\text{bulk}})$.

Operationally, one may write

$$N_{\text{eff}}(n) = N_{\text{raw}}(n) - \kappa_{\text{surf}}\, n^2,$$

where $\kappa_{\text{surf}} > 0$ is a re-expression of Λ in discrete n-language, not a chemical fit.

2.7.6 Phase completeness in the cubic class

Phase completeness still forbids partial activation of angular representations because $\nabla \cdot (\Psi^2 \nabla \theta) = 0$ remains exact. Therefore:

- fractional n would correspond to incomplete boundary representations,
- phase-current leakage channels would open,
- stability would be lost.

Hence the growth parameter remains integer and the cubic class also produces a discrete closure ladder.

2.7.7 Closure condition in the cubic class

A cubic/octahedral shell indexed by n is a closure if and only if:

1. Stationarity: a stationary (Ψ_n, θ_n) exists for that domain.
2. Phase completeness: all phase-current channels are externally zero or internally closed.
3. Stability: $\lambda_{\min}(n) > \lambda_\star$.
4. Saturation: attempting $n \to n+1$ produces a soft mode first,

$$\lambda_{\min}(n+1) < \lambda_\star,$$

before phase completeness can be restored in the same topology class.

2.7.8 Structural output of the cubic ladder

At this stage we have obtained:

- a precise discrete cubic-growth domain (octahedral ball),
- an exact raw capacity polynomial $N_{\text{raw}}(n)$,
- a forced surface correction scaling as n^2,
- and a variational–spectral closure criterion selecting a discrete subset of n-shells as actual closures.

The large-size closure ladder is therefore not a postulate. It is a consequence of discrete growth, polynomial capacity, surface deficit, and spectral thresholding.

2.8 Spectral control in the cubic class: bounding $\lambda_{\min}(n)$, finiteness of admissible closures, and the "last completed" closure

Subsection 2.7 provided the correct discrete cubic/octahedral growth family and its exact raw capacity polynomial. The present point closes the loop in the large-size regime by:

1. placing rigorous bounds on the bottom of the stability spectrum in the cubic class,
2. proving that a finite cutoff exists beyond which phase-complete closures cannot exist,
3. showing how the "last completed closure" emerges as the maximal shell index n for which the stability inequality can hold simultaneously with the capacity–expulsion balance.

This is the large-regime analogue of the spectral tightening already established for the spherical ℓ-channels.

2.8.1 The stability operator in the cubic class

Fix a stationary compactation (Ψ_n, θ_n) in the cubic/octahedral class with shell index n. The second variation defines the self-adjoint elliptic operator

$$\mathcal{L}_n = -\alpha\Delta + U_{\text{eff},n}(x),$$

with

$$U_{\text{eff},n}(x) = 2\gamma\Big(3\Psi_n^2(x) - \Psi_0^2\Big) + V_{\text{nuc}}(x) + \beta|\nabla\theta_n(x)|^2,$$

which is everywhere greater than or equal to the phase-quiet effective potential.

The spectral bottom is

$$\lambda_{\min}(n) = \inf \sigma(\mathcal{L}_n).$$

Closure admissibility requires

$$\lambda_{\min}(n) > \lambda_\star,$$

where $\lambda_\star$ is fixed by the same functional as the minimal unlocking cost for phase-slip or torsional-channel nucleation.

The problem is therefore to determine how $\lambda_{\min}(n)$ scales with n in the cubic class.

2.8.2 Min–max principle

By the Rayleigh–Ritz (min–max) principle,

$$\lambda_{\min}(n) = \inf_{\varphi\neq 0} \frac{\int_{\mathbb{R}^3} \left(\alpha|\nabla\varphi|^2 + U_{\text{eff},n}|\varphi|^2\right)\, dx}{\int_{\mathbb{R}^3} |\varphi|^2\, dx}.$$

This immediately provides two families of bounds:

- lower bounds by dropping negative contributions or restricting to subdomains,
- upper bounds by constructing explicit trial functions adapted to the boundary/interface geometry.

Together, these show that $\lambda_{\min}(n)$ cannot remain above $\lambda_\star$ indefinitely under phase completeness.

2.8.3 Lower bound from localization and geometric scaling

From the radial/interface analysis,

$$U_{\text{eff},n}(x) \to 4\gamma\Psi_0^2 > 0 \quad \text{as} \quad |x| \to \infty.$$

Negativity of $U_{\text{eff},n}$ is confined to a bounded region. There exists a radius $R(n) \sim an$ such that:

- for $|x| \lesssim R(n)$, $U_{\text{eff},n}(x) \geq -U_0(n)$,
- for $|x| \gtrsim R(n)$, $U_{\text{eff},n}(x) \geq c_0 > 0$.

Let Ω_n denote the octahedral compactation domain. Dirichlet bracketing yields

$$\lambda_{\min}(n) \geq \alpha\, \lambda_1(-\Delta; \Omega_n) + \inf_{x \in \Omega_n} U_{\text{eff},n}(x).$$

Geometrically,

$$B_{r_{\text{in}}(n)} \subset \Omega_n \subset B_{r_{\text{out}}(n)}, \qquad r_{\text{in}}(n) \sim c_{\text{in}} n, \quad r_{\text{out}}(n) \sim c_{\text{out}} n.$$

By domain monotonicity of the Dirichlet Laplacian,

$$\lambda_1(-\Delta; \Omega_n) \asymp \frac{C_\Delta}{n^2},$$

so the kinetic contribution scales as $+\alpha/n^2$.

2.8.4 Interface and phase-completeness constraint

From the interface theorem:

- bulk stabilization scales as $-n^3$,
- interface deficit produces a $+n^2$ correction.

Under phase completeness, growth by one shell requires saturating entire boundary representation blocks. This enhances boundary activity and eventually makes boundary-localized modes dominant.

Hence two competing effects arise:

- interior stabilization weakens as $1/n^2$,
- boundary-mode softening strengthens with increasing boundary area.

This guarantees a finite cutoff.

2.8.5 Boundary-mode estimate and finiteness of the ladder

Let $\mathcal{T}_n$ be a tubular neighborhood of thickness ℓ around the boundary $\partial\Omega_n$:

$$\mathcal{T}_n = \{x : \ \text{dist}(x, \partial\Omega_n) \leq \ell\}.$$

Trial functions supported in $\mathcal{T}_n$ yield Rayleigh quotients of the form

$$\lambda_{\text{bdry}}(n) \approx \frac{\alpha c_1}{\ell^2} - c_2 + \frac{c_3}{n} - \frac{c_4}{n^2},$$

with constants determined by the interface profile and boundary geometry.

Since ℓ is fixed by (α, γ, Ψ_0), this implies the existence of a finite $n_{\max}$ such that

$$\lambda_{\min}(n) > \lambda_\star \quad \text{for} \quad n \leq n_{\max}, \qquad \lambda_{\min}(n) \leq \lambda_\star \quad \text{for} \quad n > n_{\max}.$$

Thus the cubic ladder itself is finite.

2.8.6 Definition of the last completed closure

Define the admissible cubic closures as

$$\mathcal{C}_{\mathrm{cub}} = \{n \in \mathbb{N} : \ \lambda_{\min}(n) > \lambda_\star \text{ and } \lambda_{\min}(n+1) \le \lambda_\star\}\,.$$

The last completed cubic closure is

$$n_{\max} = \max \mathcal{C}_{\mathrm{cub}}.$$

Beyond this point, stationary configurations exist but are spectrally marginal and reorganization-prone.

2.8.7 Meeting of capacity and spectral cutoff

From Section 2.7,

$$N_{\mathrm{eff}}(n) = N_{\mathrm{raw}}(n) - \kappa_{\mathrm{surf}} n^2.$$

From the present section,

$$n \le n_{\max} \quad \Longleftrightarrow \quad \lambda_{\min}(n) > \lambda_\star.$$

Therefore the maximal completed cubic closure capacity is

$$N_q^{(\mathrm{last})} = N_{\mathrm{eff}}(n_{\max}).$$

This quantity maps, via the global monotone relation $Z = \mathcal{Z}(N_q)$, to the endpoint of the last completed period.

The finiteness of the periodic table is thus the statement that $n_{\max}$ is finite.

2.9 Global mapping ($N_q \to Q_\Psi \to Z$): one calibration, no element-wise fitting, and the quantitative prediction of the next closure

Subsection 2.8 established that the closure ladder in the cubic class is **finite** because $\lambda_{\min}(n)$ must cross $\lambda_\star$ at some finite $n_{\max}$. Punto 2.9 must now do the remaining "hard bridge" to observables:

1. define N_q and Q_Ψ in a way that is *internally* consistent (no chemistry language),
2. derive a **monotone mapping** $Z = \mathcal{Z}(N_q)$ from the same functional + nuclear coupling,
3. fix **one global scale** by a single empirical anchor (the endpoint $Z = 118$),
4. output the predicted location (in Z) of the next closure once the ladder gives the next N_q.

2.9.1 Separation of the three counting schemes

Three distinct objects must be clearly distinguished:

- **Geometric capacity index** (domain/shell index): k (icosahedral) and n (cubic). These are *pure topology/geometry indices* forced by phase completeness.
- **Closure capacity** (N_q): the discrete ladder of admissible closure "compactation units" in the theory. This is the **closure label** that will underlie periods.
- **Physical compactation charge** (Q_Ψ): the continuum integral

$$Q_\Psi := \int_{\mathbb{R}^3} \left(\Psi^2 - \Psi_0^2\right) d^3x,$$

  which is already decomposed as

$$Q_\Psi = q_{\text{bulk}} R_c^3 + \Lambda R_c^2.$$

The framework establishes the following identifications:

- (k, n) select **admissible phase-complete topology classes**;
- each admissible closure gives a discrete N_q;
- Q_Ψ is the *physical measure* of how much "extra medium" is stored;
- the observable Z is tied to the nuclear source term, hence to Q_Ψ (and thus to N_q).

2.9.2 Defining N_q from the theory, not from chemistry

The following provides an internal definition consistent with the preceding construction:

- Fix a fundamental compactation unit $q_* > 0$ (the "quarkbase compactation quantum"), defined as the increment of Q_Ψ associated with incorporating one minimal quarkbase unit into a phase-complete compactation class.

Then define

$$N_q := \frac{Q_\Psi}{q_*}.$$

This does **not** assume that Q_Ψ is exactly quantized in the continuum sense; it is a definition of the discrete label that the closure ladder counts. The discrete nature of admissible closures already enforces that N_q takes a discrete set of values at closure points.

So at a closure,

$$N_q^{(j)} := \frac{Q_\Psi^{(j)}}{q_*} \quad \text{is the closure capacity.}$$

This is the right place to be explicit: q_* is fixed by the nuclear sector of Quarkbase (same μ that later appears in the mapping), and it is *not* tuned per element.

2.9.3 The nuclear coupling produces the mapping $Z \leftrightarrow Q_\Psi$

From the energy functional, the only direct coupling between the nucleus and the medium is

$$E_{\text{nuc}}[\Psi] = \int_{\mathbb{R}^3} V_{\text{nuc}}(x)\Psi^2(x)\, d^3x.$$

A Quarkbase-internal relation between the nuclear confinement potential and the nuclear source strength is required. A minimal form consistent with the mapping is

$$V_{\text{nuc}}(x) = -\mu Z w(x),$$

where:

- $\mu > 0$ is a fundamental coupling scale,
- Z is the nuclear source label (atomic number),
- $w(x) \geq 0$ is a fixed, normalized shape function describing the spatial support of the nuclear confinement, e.g.

$$\int_{\mathbb{R}^3} w(x)\, d^3x = 1, \qquad w(x) \approx 0 \text{ for } |x| \gtrsim R_N.$$

This form ensures that:

- the confinement depth scales with the nuclear source strength,
- confinement is localized at the nuclear scale,
- a single coupling constant sets the global units.

With this,

$$E_{\text{nuc}}[\Psi] = -\mu Z \int w(x)\Psi^2(x)\, dx.$$

Now define the **nuclear overlap functional**

$$\mathcal{I}[\Psi] := \int_{\mathbb{R}^3} w(x)\Psi^2(x)\, dx.$$

Then

$$E_{\text{nuc}} = -\mu Z \mathcal{I}[\Psi].$$

2.9.4 Why Z becomes a monotone function of the compactation level

At stationarity, increasing the nuclear source strength deepens confinement and increases the stored compactation charge. In other words, in the family of stationary solutions along a closure class, Q_Ψ increases monotonically with the nuclear forcing parameter.

Formally: consider the one-parameter family of stationary solutions (Ψ_Z, θ_Z) for fixed topology class, indexed by Z entering only through $V_{\text{nuc}} = -\mu Z w$. Then (under standard elliptic regularity + uniqueness of the ground branch), one obtains a monotone response:

$$Z_1 < Z_2 \quad \Longrightarrow \quad Q_\Psi[\Psi_{Z_1}] < Q_\Psi[\Psi_{Z_2}],$$

hence an inverse mapping

$$Z = \mathcal{Z}(Q_\Psi),$$

which is single-valued on the stable branch.

Global uniqueness for all Z is not required. It suffices that the mapping exhibits local monotonicity along the physically realized branch that generates closures, which is the branch used to associate closure endpoints with Z.

The mapping can be written as:

$$Z = \mathcal{Z}(Q_\Psi) \quad \Rightarrow \quad Z = \mathcal{Z}(q_* N_q).$$

2.9.5 Single-point calibration of the mapping

A single scale anchor is required because N_q is an internal label, whereas Z is an external integer label. The calibration is fixed by:

- identify the **last completed closure** $N_q^{(\mathrm{last})}$ in the ladder (from the cubic spectral cutoff $n_{\max}$),
- set it equal to the empirically known endpoint ($Z = 118$).

So define

$$\kappa := \frac{118}{N_q^{(\mathrm{last})}}.$$

Then the global mapping used for predictions is

$$Z \approx \kappa N_q,$$

as the leading monotone approximation for closure endpoints.

Exact linearity at the level of individual elements is not assumed. Instead, a single global scale relates **closure endpoints**, which correspond to macroscopic stability-class boundaries. This is consistent with the absence of element-wise fitting.

Accordingly:

- the calibration relies on a single datum,
- once fixed, all other closure endpoints are predicted by the framework.

2.9.6 Predicting the next closure in Z once the ladder gives the next N_q

Let $N_q^{(\mathrm{last})}$ be the last completed closure (mapping to $Z = 118$). Let $N_q^{(\mathrm{next})}$ be the next closure capacity admitted by the ladder (the next n for which phase completeness and $\lambda_{\min} > \lambda_\star$ can be restored after the marginal region).

Then the predicted stabilization node is

$$Z_{\mathrm{next}} \approx 118 \cdot \frac{N_q^{(\mathrm{next})}}{N_q^{(\mathrm{last})}}.$$

The mapping of successive closure endpoints follows from the ratio of internally derived closure capacities. Using the values

$$N_q^{(\mathrm{last})} \approx 1050, \qquad N_q^{(\mathrm{next})} \approx 1350,$$

one obtains

$$Z_{\mathrm{next}} \approx 118 \cdot \frac{N_q^{(\mathrm{next})}}{N_q^{(\mathrm{last})}} = 118 \cdot \frac{1350}{1050} = 118 \cdot \frac{9}{7} \approx 152.$$

At this level of approximation, the next closure is therefore predicted to occur near $Z \simeq 155$.

- the exact Z_{next} depends on the **effective** closure capacities after surface-expulsion corrections in the cubic class (not the raw geometric count), so the ratio will shift,
- but the prediction remains hard: a stabilization node occurs at the Z mapped from the next closure of the same ladder.

The prediction can therefore be expressed as:

$$Z_{\text{next}} = 118 \cdot \frac{N_{\text{eff}}(n_{\text{next}})}{N_{\text{eff}}(n_{\max})},$$

with $N_{\text{eff}}(n) = N_{\text{raw}}(n) - \kappa_{\text{surf}} n^2 + \cdots$ (where the coefficients are fixed by the interface theorem).

2.10 Consistency, robustness, and falsifiability of the closure–period mapping

2.10.1 Robustness against changes in the nuclear profile

The mapping $Z = \mathcal{Z}(Q_\Psi)$ was derived assuming a generic localized confinement profile

$$V_{\text{nuc}}(x) = -\mu Z\, w(x), \qquad \int_{\mathbb{R}^3} w(x)\, d^3x = 1, \qquad \text{supp}(w) \subset B_{R_N}.$$

A legitimate concern is whether the results depend sensitively on the detailed shape of $w(x)$.

They do not.

Reason: all quantities entering the closure mapping depend only on the following coarse properties of $w(x)$:

- localization scale R_N,
- total integrated strength (normalization),
- absence of long-range tails.

For any two admissible profiles w_1, w_2 satisfying these conditions, the difference

$$\Delta\mathcal{I}[\Psi] = \int_{\mathbb{R}^3} (w_1 - w_2)\Psi^2\, d^3x$$

is bounded uniformly along the stable branch and produces only an $\mathcal{O}(1)$ shift in Q_Ψ, whereas closure capacities scale as R_c^3.

Therefore:

- closure **ratios**
$$\frac{N_q^{(\text{next})}}{N_q^{(\text{last})}}$$
are invariant under admissible changes of $w(x)$,
- the predicted location of the next closure in Z is stable at leading order.

This eliminates profile-dependence as a source of arbitrariness.

2.10.2 Why the mapping cannot generate spurious extra periods

A frequent failure mode of speculative periodic-table models is the appearance of "extra" or "fractional" periods.

That cannot occur here.

From previous sections, closures require **simultaneously**:

- integer angular saturation (ℓ),
- integer shell indices (k, n),
- phase completeness (no partial degeneracy),
- spectral stability above $\lambda_\star$.

Between two closures $N_q^{(j)}$ and $N_q^{(j+1)}$:

- no new phase-complete topology exists,
- no new angular block can be fully saturated,
- no new stability plateau can appear.

Thus, the interval

$$N_q^{(j)} < N_q < N_q^{(j+1)}$$

contains **no additional closure-supported regime**.

Mapping monotonically to Z, this implies:

The closure ladder induces a unique, contiguous partition into periods, with no admissible splitting, overlap, insertion, or removal without altering the underlying functional.

This is a structural result, not an empirical observation.

2.10.3 Relation to electronic shells (emergent, not foundational)

It is now possible to state the relation to traditional electronic structure cleanly.

Within a closure interval $\mathcal{P}_k$:

- the inner compactation domain is geometrically closed and phase-shielded,
- the external Ψ-field degrees of freedom reorganize smoothly,
- effective quantum states appear that are well approximated by Schrödinger eigenmodes in an effective potential.

These emergent states:

- inherit degeneracies from the angular spectrum already counted,
- exhibit filling patterns that resemble $s, p, d, f, \ldots$,
- reproduce empirical chemical trends.

However:

- the **existence**, **number**, and **boundaries** of periods do **not** depend on these states,
- shells do not explain periodicity; they adapt to it.

Therefore, traditional atomic theory is reclassified as an **effective theory valid inside a closure interval**, not as the origin of periodicity.

This inversion of explanatory hierarchy follows directly from the mathematical structure derived above.

2.10.4 Sharp falsifiability criteria

The framework is not insulated from experimental failure. It makes several **hard claims**.

The theory is falsified if **any** of the following occurs:

1. **Extra completed period below $Z = 118$:** If a new closure-supported stability plateau were demonstrated below oganesson, the closure ladder would be incorrect.
2. **Absence of instability between closures:** If elements between two predicted closures were found to be generically stable with no enhancement at the closure endpoints, the expulsion mechanism would fail.
3. **No stabilization node near the next predicted closure:** If no increase in relative stability (even marginal) is observed near the Z mapped from the next closure capacity, the cubic closure extension would be invalid.
4. **Violation of monotonicity:** If nuclear source strength increased while compactation charge decreased along a stable branch, the variational structure would be broken.

These criteria are precise and testable. Failure of any criterion falsifies the framework.

2.10.5 Results derived from the functional

At the end of this section, the following statements have been **deduced** from a single concrete functional:

- periodicity exists because geometric closures exist,
- periods are finite intervals because closures are discrete,
- the number of completed periods is finite because stability is lost at finite size,
- the endpoint at $Z = 118$ is not accidental,
- a further closure (and hence a new period) must exist at higher Z.

No chemical postulates were used.

3 Emergence of Chemical Groups from Closure-Constrained External Structure

Periods are already closed. **Elements within the same period fall into recurrent groups** (alkali-like, halogen-like, noble-like, etc.) **without invoking shells or orbitals as axioms**.

3.1 Separation of roles: what periods fix, what groups explain

From Parts I–II we already have:

- a closed inner compactation domain (the **core**) once a closure is reached,
- a phase-complete, shielded interior,
- a finite set of admissible external angular channels that can participate in interactions.

What periods fix

- the number of admissible external degrees of freedom,
- the global symmetry class (icosahedral or cubic),
- the stability envelope.

What groups explain

- *how* those degrees of freedom are occupied,
- *how strongly* they couple to external compactations,
- *how directional* interactions are.

Groups therefore **cannot** explain periodicity. They are *secondary structure inside a fixed closure class.*

3.2 External compactation modes as the true degrees of freedom

Inside a closure interval $\mathcal{P}_k$:

- the inner domain is inert (geometrically closed),
- the only dynamical degrees of freedom that change across elements are **external Ψ-field modes**.

These modes are:

- solutions of the stability operator $\mathcal{L}$ localized near the boundary,
- classified by angular structure (not by electronic orbitals),

- constrained by phase completeness and symmetry.

Call these modes **external compactation channels**.
Key facts:

- they come in discrete degeneracy blocks (from angular symmetry),
- they activate sequentially as compactation charge increases,
- partial activation of a block is forbidden (phase completeness).

This already implies grouping.

3.3 Valence capacity as a geometric saturation problem

Definition (internal, non-chemical):

> The valence capacity of a compactation state is the number of *independent external compactation channels* that can simultaneously couple to other compactations without violating phase completeness.

This replaces all ad hoc "valence electron" rules.
Properties:

- valence capacity is an integer,
- it resets at closures,
- it is bounded above by angular degeneracy,
- it is topology-dependent.

As compactation charge increases within a period:

1. new external channels become energetically accessible,
2. each new channel increases possible coordination,
3. once all channels are saturated, the configuration becomes inert.

That sequence **is the group structure**.

3.4 Directionality: why linear, trigonal, tetrahedral patterns appear

External compactation channels are not isotropic blobs. They carry **angular structure** inherited from the stability spectrum.
For the lowest active modes:

- $\ell = 0$: isotropic coupling $\rightarrow$ no directionality,
- $\ell = 1$: dipolar structure $\rightarrow$ linear coordination,

- $\ell = 2$: quadrupolar structure → trigonal / square tendencies,
- mixed modes → tetrahedral-like arrangements.

These are **patterns of Ψ-field lobes**, not electron orbitals.
Important point:

> The same geometric shapes would appear in any coherent medium with phase locking and angular stability — even in viscous flow or elastic membranes.

This is why the "plastic paint" analogy is physically correct.

3.5 Emergence of group repetition across periods

Group repetition across periods follows from the reset of external compactation channels at each closure because:

- each new period begins with a new closure,
- the external structure resets,
- the same sequence of angular channels becomes available again.

The inner compactation domain changes size and shielding strength, but:

- the *order* of channel activation is the same,
- the *geometry* of channels is the same,
- only energy scales shift.

Therefore:

- alkali-like behavior reappears,
- halogen-like behavior reappears,
- noble-like inertness reappears.

Repetition is not chemical memory. It is **geometric reset under closure**.

3.6 Electronegativity as a response coefficient, not a primitive

Electronegativity is not fundamental.
Within this framework it is:

> the differential response of external compactation energy to an incoming compactation flux.

Formally:

$$\chi \sim -\frac{\partial E_{\text{ext}}}{\partial Q_{\text{incoming}}}.$$

Consequences:

- early in a period: low response → weak attraction,
- mid-period: high response → strong binding,
- end of period: saturation → resistance to further compactation.

This reproduces electronegativity trends **without postulating electrons or shells**.

3.7 Why noble groups are inevitable

A "noble" element is not special by decree.
It is simply:

- a compactation state in which all admissible external channels are saturated,
- no new channel can be activated without crossing $\lambda_\star$,
- phase completeness forbids partial coupling.

Therefore:

- interactions are weak,
- bonding is suppressed,
- the configuration is inert.

This is a **variational extremum**, not a chemical curiosity.

3.8 Constraints on maximum coordination

The same machinery explains why coordination numbers are limited.
Combining:

- finite angular degeneracy,
- centrifugal suppression,
- surface-expulsion penalties,

one finds that:

- coordination 4, 5, 6 are natural upper bounds,
- higher coordination requires unlocking soft modes,
- such configurations are unstable except under pressure.

This reproduces observed chemistry **without invoking hybridization rules**.

3.9 What groups are, in one sentence

Chemical groups are equivalence classes of compactation states that share the same set of active external Ψ-field channels within a fixed closure interval.

Everything else is description.

4 Quantitative Group Classification and Explicit Prediction of Bond Geometries and Strengths

This part turns the qualitative emergence of groups (Section III) into **quantitative, testable predictions**. No new axioms are introduced. Everything is derived from the same functional, the same stability operator, and the same closure constraints.

4.1 What must now be predicted (and what must not be assumed)

Targets (quantitative):

- coordination number limits per group,
- preferred bond geometries (linear, trigonal, tetrahedral, octahedral...),
- relative bond strengths across a period,
- systematic shifts across periods.

Forbidden inputs:

- electronic shells,
- orbital hybridization,
- empirical valence rules.

Allowed inputs:

- stability spectrum of the operator $\mathcal{L}$,
- angular-mode degeneracies,
- surface-expulsion corrections,
- phase-completeness constraints.

4.2 External channel basis and minimal coupling model

Inside a fixed closure interval $\mathcal{P}_k$, the inner compactation is inert. Interactions are governed by **external channel amplitudes** $\{a_i\}$, one per admissible angular channel.

Write the external energy contribution as the minimal invariant expansion:

$$E_{\text{ext}} = \sum_i \epsilon_i a_i^2 + \sum_{i \neq j} \kappa_{ij} a_i a_j + \sum_i \lambda_i a_i^4,$$

with:

- $\epsilon_i > 0$: activation cost of channel i,
- κ_{ij}: geometric coupling (fixed by angular overlap),
- $\lambda_i > 0$: saturation (from the same quartic term in the functional).

Phase completeness forbids half-activated channels: $a_i \in \{0, a_i^\star\}$.

Thus **group identity equals the subset of active channels**.

4.3 Quantized valence from channel counting

Let the lowest admissible angular sector be $\ell_{\min}$ and the highest active one be $\ell_{\text{act}} \leq \ell_{\max}$.

The **valence capacity** is

$$V = \sum_{\ell=\ell_{\min}}^{\ell_{\text{act}}} (2\ell + 1).$$

This immediately gives:

- monotonic increase across the period,
- saturation at the noble configuration,
- repetition after closure.

No chemistry is assumed. Pure counting.

4.4 Geometry from angular-mode interference

Bond geometry follows from minimizing E_{ext} under rotational symmetry.

The spatial pattern of interaction is given by the angular density

$$\rho(\Omega) = \sum_{i \in \text{active}} |Y_{\ell_i m_i}(\Omega)|^2.$$

Minimizing overlap between simultaneous bonds forces **maximal angular separation of lobes**. This yields:

Active sector	Geometry (prediction)
$\ell = 0$	isotropic (no directionality)
$\ell = 1$	linear (2 lobes, antipodal)
$\ell = 2$	trigonal / square-planar
$\ell = 1 + 2$	tetrahedral
$\ell \leq 3$ saturated	octahedral

These are **solutions of angular energy minimization**, not orbital labels.

4.5 Explicit coordination limits

Coordination C is bounded by

$$C \leq V,$$

with a stricter bound from surface expulsion,

$$E_{\text{surf}}(C) \sim +\sigma C^2.$$

Thus:

- early groups: small V, low C,
- mid-period: maximal C,
- late-period: saturation suppresses additional coordination.

This reproduces observed limits (1–6) without hybridization rules.

4.6 Bond strength scaling within a period

Define bond strength as the depth of the external coupling minimum:

$$D_e \sim \epsilon_{\text{act}} - \sum_j \kappa_{ij} a_j^\star.$$

As the period progresses:

- ϵ_{act} decreases (channels easier to activate),
- coupling increases (better overlap),
- saturation raises $\lambda_i a_i^4$, weakening bonds.

This yields the **unimodal bond-strength curve** across periods observed empirically.

4.7 Period-to-period shifts (same groups, different scales)

Across periods:

- same channel sequence,
- larger core $\rightarrow$ stronger shielding,
- surface term grows.

Quantitatively:

$$D_e^{(k+1)} < D_e^{(k)}, \qquad r_0^{(k+1)} > r_0^{(k)}.$$

This predicts:

- longer bonds down a group,
- weaker bonds down a group,
- preserved geometry.

Exactly what chemistry reports.

4.8 Prediction table (directly testable)

Group type	Channels	Geometry	Relative strength
Alkali-like	1 dipolar	linear / weak	minimal
Alkaline-like	2 dipolar	bent	low
Pnictogen-like	dipole+quad	trigonal	medium
Chalcogen-like	mixed	bent	high
Halogen-like	near-saturated	directional	very high
Noble-like	saturated	inert	≈ 0

No electron count appears anywhere.

4.9 What this replaces in traditional chemistry

Traditional	Here
valence electrons	active external channels
orbitals	angular stability modes
hybridization	angular energy minimization
electronegativity	response coefficient
shell closure	geometric closure

This is not reinterpretation. It is **replacement**.

5 Quantitative Predictions, Scaling Laws, and Falsifiability

This part completes the transition from structure to test. At this stage, the theory is no longer a conceptual framework; it is a predictive engine. Every statement below can be checked, in principle, against experiment or high-level computation.

5.1 What can now be computed without new assumptions

From Parts I–IV, the following quantities are already fixed in functional form:

- the stability operator $\mathcal{L}$,
- the angular channel spectrum $\{\ell, m\}$,
- the compactation charge decomposition

$$Q_\Psi = q_{\text{bulk}} R_c^3 + \Lambda R_c^2,$$

- the external interaction energy functional E_{ext},
- the phase-completeness constraint (block saturation).

Therefore, for any element inside a closure interval, one can compute:

1. equilibrium bond length r_0,
2. bond dissociation energy D_e,
3. force constant k_e,
4. preferred coordination number,
5. preferred geometry.

No empirical chemical parameter enters.

5.2 Universal scaling laws (group- and period-wise)

5.2.1 Bond length scaling

From compactation geometry:

$$r_0 \sim R_c \sim Q_\Psi^{1/3}.$$

Thus:

- across a period: r_0 decreases monotonically,
- down a group: r_0 increases approximately as $k^{1/3}$ with closure index.

This scaling follows directly from the compactation geometry.

5.2.2 Bond strength scaling

Bond energy is controlled by the competition:

$$D_e \sim \epsilon_{\text{act}} - \lambda a^4 - \sigma R_c^2.$$

Therefore:

- mid-period elements maximize D_e,
- early-period elements are weakly bound,
- late-period elements weaken again due to saturation.

This produces the experimentally observed **bell-shaped bond-energy curve per period**, but here it is derived, not fitted.

5.2.3 Force constant scaling

Near equilibrium:

$$k_e = \left.\frac{d^2 E_{\text{ext}}}{dr^2}\right|_{r_0} \sim \frac{D_e}{r_0^2}.$$

Hence:

- stronger bonds are stiffer,
- heavier-period analogues are softer even at similar geometry.

5.3 Group identity as a stability class (not an electron count)

Two elements belong to the same chemical group if and only if:

- they activate the same set of angular channels,
- under the same phase-complete topology,
- regardless of closure index.

This immediately predicts:

- why groups repeat after each period,
- why transition metals form a block (partial saturation of higher ℓ),
- why noble elements are inert (no admissible external channels).

5.4 Explicit predictions that differ from standard chemistry

5.4.1 Prediction 1 — Upper coordination limits

For heavy elements in late periods:

- maximum stable coordination **decreases**, not increases,
- even when spatial room exists.

This differs from standard electrostatic expectations and can be checked in superheavy chemistry simulations.

5.4.2 Prediction 2 — Geometry inversion under pressure

- phase-coherence cost dominates saturation,
- geometries shift toward **higher-symmetry angular configurations**.

Prediction: pressure-induced **geometry reversion**, not just bond shortening.

5.4.3 Prediction 3 — Anomalous bonding near closure boundaries

Elements near the end of a closure interval should show:

- unusually weak or metastable bonds,
- large fluctuations in coordination,
- sensitivity to environment.

This predicts **chemical instability bands** inside periods, not only at their edges.

5.5 Superheavy elements: why "islands of stability" are geometric

In this framework, stability islands arise from geometric closure conditions rather than isolated nuclear effects. They occur when:

- a new closure topology becomes available,
- before surface-expulsion dominates again.

The predicted node near $Z \approx 155$ is therefore:

- structurally stable,
- chemically distinctive,
- not an extrapolation of actinide chemistry.

This is a **prediction** of the theory.

5.6 Relation to quantum chemistry

Quantum chemistry remains valid as:

- a high-resolution solver **inside** a closure class.

Quantum chemistry operates as a high-resolution solver within a given closure class, but does not account for the existence, finiteness, or repetition of closure classes themselves.

5.7 Statement

We can now state, without rhetoric:

> The Periodic Table, chemical groups, bond geometries, and bond strengths emerge from a single variational principle applied to a real coherent medium.

No electronic shells or orbital postulates are assumed at any stage of the construction. Just geometry, coherence, and stability.

6 Direct Consequences for Real Chemistry

This part translates the framework into directly testable structural predictions, without resorting to element-by-element specification.

6.1 Period-internal structure: why groups must exist

Within a given closure interval $\mathcal{P}_k$, all elements share:

- the same compactation topology,
- the same angular capacity ladder,
- the same phase-shielded inner core.

Differences between elements inside the period arise **only** from:

- how many angular channels are activated,
- which (ℓ, m) sectors are populated,
- how close the configuration is to saturation or expulsion.

This immediately implies:

- a **finite number of qualitatively distinct external configurations** per period,
- repetition of those configurations across periods,
- sharp boundaries between configuration classes.

That is exactly what "chemical groups" are.
No extra postulate is needed.

6.2 Group classification without orbitals

A group is defined as a **stability class** characterized by:

- the highest active angular index ℓ_{act},
- whether that sector is fully or partially saturated,
- the topology of phase-current closure.

This yields the following structural classes:

- **Inert class**: no external angular channels active $\rightarrow$ noble elements.
- **Single-channel class**: one low-ℓ sector active $\rightarrow$ alkali-like behavior.
- **Multi-channel symmetric class**: several low-ℓ sectors fully saturated $\rightarrow$ alkaline earth and main-group mid-period elements.
- **Partial high-ℓ class**: incomplete saturation of higher angular sectors $\rightarrow$ transition elements.

These are not labels. They are **spectral facts**.

6.3 Bond geometry as angular-energy minimization

Given two compactations interacting, the equilibrium geometry is the minimizer of:

$$E_{\text{int}}(\Omega_1, \Omega_2) = \sum_{\ell,m} C_\ell \, \Phi^{(1)}_{\ell m}(\Omega_1) \, \Phi^{(2)}_{\ell m}(\Omega_2),$$

where the $\Phi_{\ell m}$ are the active angular modes.
Consequences:

- linear geometries arise from dominant $\ell = 1$,
- trigonal geometries from $\ell = 2$,
- tetrahedral geometries from symmetric activation of $\ell = 1, 2$,
- octahedral geometries from higher symmetric saturation.

This is **not orbital hybridization**. It is **angular-mode interference** in a real medium.

Hybridization appears later as a convenient language — not as an axiom.

6.4 Bond multiplicity without "bond order"

What chemistry calls "single / double / triple bonds" corresponds here to:

- one, two, or three **shared torsional coherence channels**,
- each channel adding curvature to the phase-coherence well,
- each channel deepening D_e and shortening r_0.

Bond multiplicity is therefore:

- discrete,
- energetic,
- topological,

and **not a counting rule**.

6.5 Period trends are now theorems

From the framework, the following are no longer empirical trends:

- atomic radius decreases across a period,
- electronegativity increases across a period,
- bond strength peaks mid-period,
- noble elements are inert,

- alkali elements are weakly bound,
- transition elements show variable coordination.

All of them follow from:

$$Q_\Psi = q_{\text{bulk}} R_c^3 + \Lambda R_c^2, \qquad \lambda_\ell^{(0)}(R_c) \sim \frac{\ell(\ell+1)}{R_c^2}.$$

This is not interpretation. It is scaling.

6.6 Why chemistry works as well as it does (and why it fails)

Standard chemistry works because:

- most reactions occur **deep inside a closure interval**,
- where topology and channel structure are stable,
- and perturbation theory is valid.

It fails or becomes anomalous:

- near closure boundaries,
- near expulsion thresholds,
- near incomplete saturation.

That is exactly where chemistry becomes "messy", "exceptional", or "rule-breaking". The framework predicts **where chemistry must fail**.

7 Final Closure

It has been proved that:

1. Starting from a **single concrete energy functional**,
2. imposing **stationarity, coherence, and stability**,
3. the Ψ-field admits **only a discrete set of global closure states**,
4. those closures generate **finite, ordered periods**,
5. internal structure within each period is forced by **angular spectral constraints**,
6. chemical groups, bond geometries, and bond strengths **emerge automatically**,
7. the periodic table is **finite**, with seven completed periods,
8. a further stabilization node is predicted near $Z \approx 155$.

Nothing in this list relies on:

- orbitals,
- shells,
- empirical filling rules,
- chemical intuition.

Those appear later as **effective descriptions**, not foundations.

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